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iome Useful Probability Distributions

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I. Omura and T. Kailath

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Technical Report No. 7050-6

Prepared under
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U S. Air Force and the U.S. Navy
(Office of Naval Research)

SYSTEMS THEORY LABORATORY

STANFORD ELECTRONICS LABORATORIES

STANFORD UNIVERSITY · STANFORD, CALIFORNIA

SEL-65-079

SOME USEFUL PROBABILITY DISTRIBUTIONS

by

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Systems Theory Laboratory
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ABSTRACT

We present a compilation of probability density functions, distribution functions, and characteristic functions for several arithmetic combinations of Gaussian random variables. These functions, the distribution functions in particular, need to be known for the evaluation of the error probabilities of many communication systems. We endeavour, as far as possible, to express these functions in "closed form" in terms of well known (and tabulated) transcendental functions. Our purpose is to have a convenient list of canonical forms which will be useful when working with arithmetic combinations of Gaussian random variables.

Since distribution functions are the most useful, they are listed first. In the second half the density functions, characteristic functions, and some moments are listed. Both halves follow the same outline so that one can easily locate corresponding functions relating to the same random variable. Since many distribution functions could not be found, in the list of distribution functions there are dashes, --, to indicate these yet unknown functions.

Some miscellaneous results are given in Appendix A. The transcendental functions are defined in Appendix B along with some useful relationships.

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OUTLINE TWO - DENSITY FUNCTIONS AND CHARACTERISTIC FUNCTIONS

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OUTLINE THREE - APPENDIXES AND REFERENCES

DEFINITIONS AND NOTATION

Throughout this report capital letters will represent vectors or matrices with the exception of F, G, and transcendental functions. $F(\cdot)$ and $G(\cdot)$ are the only letters used to represent distribution functions of random variables with $f(\cdot)$ and $g(\cdot)$ representing the corresponding density functions. $\psi(\mu)$ will denote all characteristic functions. Letters x and y are used only for Gaussian random variables and $\underline{X}$ and $\underline{Y}$ are the only letters denoting Gaussian vectors.

$\underline{X}$ is always a column vector with independent equal variance Gaussian random variables as components. That is, for $\underline{X}$ n-dimensional

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where $\{x_k\}_{k=1}^n$ are independent Gaussian random variables with equal variances, σ^2. If $E\underline{X} = \underline{A}$ then this class of Gaussian vectors will be denoted as $N_n(\underline{A},\sigma^2)$. Hence $\underline{X} \in N_n(\underline{A},\sigma^2)$ means that $\underline{X}$ is an n-dimensional Gaussian vector with independent components and with common variance σ^2 and mean $E\underline{X} = \underline{A}$.

Only Gaussian vectors of the same order are allowed to be statistically dependent. If $\underline{X}^{(1)} \in N_n\left(\underline{A},\sigma_1^2\right)$ and $\underline{X}^{(2)} \in N_n\left(\underline{B},\sigma_2^2\right)$ are dependent then we mean dependence in the following manner.

$$\underline{X}^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix}, \quad \underline{X}^{(2)} = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ \vdots \\ x_n^{(2)} \end{bmatrix}; \quad \underline{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

and

$$E\left(x_j^{(1)} - a_j\right)\left(x_k^{(2)} - b_k\right) = \begin{cases} 0 & j \neq k \\ \rho\sigma_1\sigma_2 & j = k \end{cases} \qquad j,k = 1,2,\ldots,n$$

That is, only components of $X^{(1)}$ and $X^{(2)}$ having identical subscripts can be correlated. We shall summarize this dependence by the matrix.

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}; \qquad |\rho| < 1$$

$\underline{M}$ is the common covariance matrix of the vector-pair components. We shall also define

$$\underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_1^2(1-\rho^2)} & -\frac{\rho}{\sigma_1\sigma_2(1-\rho^2)} \\ -\frac{\rho}{\sigma_1\sigma_2(1-\rho^2)} & \frac{1}{\sigma_2^2(1-\rho^2)} \end{bmatrix}$$

Clearly if $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$ are independent then $\rho = 0$.

If $\underline{X} \in N_n(\underline{0},\sigma^2)$, we shall define

$$r = \|\underline{X}\| = \left(\sum_{k=1}^{n} x_k^2\right)^{1/2}$$

as the <u>Rayleigh variable of order n</u>. r^2 is essentially the <u>central chi square variable of order n</u>.

If $\underline{X} \in N_n(\underline{A}, \sigma^2)$ we shall define

$$a = \|\underline{A}\| = \left(\sum_{k=1}^{n} a_k^2 \right)^{1/2}$$

as the norm of the mean and

$$\nu = \|\underline{X}\| = \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2}$$

as the <u>Rice variable of order n</u>. ν^2 is essentially the <u>noncentral chi square variable of order n</u>. Throughout this paper r and ν will denote only these random variables.

We shall define an inner product for $\underline{X}^{(1)} \in N_n\left(\underline{A}, \sigma_1^2\right)$ and $\underline{X}^{(2)} \in N_n\left(\underline{B}, \sigma_2^2\right)$ as

$$\left(\underline{X}^{(1)}, \underline{X}^{(2)}\right) = \sum_{k=1}^{n} x_k^{(1)} x_k^{(2)}$$

For any positive real number, c, we define [c] as

$$[c] = \min_n \{n; \; n \geq c; \; n \text{ is an integer}\}$$

If z is a random variable and $f(\cdot)$ its density function, we shall write its (cumulative) distribution function as

$$F(z) = \int_{-\infty}^{z} f(t)\, dt$$

and its characteristic function as

$$\psi_z(\mu) = \int_{-\infty}^{\infty} e^{i\mu r} f(r)\, dr , \quad i = \sqrt{-1}$$

ACKNOWLEDGMENT

The authors wish to thank Dr. Robert Price for his helpful suggestions and encouragement in preparing this report.

I. FUNDAMENTAL VARIABLES

A. GAUSSIAN

$$x \in N_1(a,\sigma^2)$$

$$F(x) = \begin{cases} \frac{1}{2} - \operatorname{erf}\left(\frac{a - x}{\sqrt{2\sigma^2}}\right) & x \leqq a \\ \frac{1}{2} + \operatorname{erf}\left(\frac{x - a}{\sqrt{2\sigma^2}}\right) & x > a \end{cases}$$

[Ref. 5, p. 136]

B. RAYLEIGH

$$\underline{X} \in N_n(\underline{0},\sigma^2); \qquad r = \|\underline{X}\|$$

1. n = 1

$$F(r) = \operatorname{erf}\left(\frac{r}{\sqrt{2\sigma^2}}\right) \qquad r \geqq 0$$

[Ref. 5, p. 136]

2. n = 2

$$F(r) = 1 - e^{-r^2/2\sigma^2} \qquad r \geqq 0$$

3. n = 2k (even)

$$F(r) = 1 - e^{-r^2/2\sigma^2} \sum_{j=0}^{k-1} \frac{1}{j!}\left(\frac{r^2}{2\sigma^2}\right)^j \qquad r \geqq 0$$

[Ref. 5, p. 134]

4. n

— — — —

C. RICE

$$\underline{X} \in N_n(\underline{A}, \sigma^2), \qquad a = \|\underline{A}\|, \qquad \nu = \|\underline{X}\|$$

1. n = 1

$$F(\nu) = \begin{cases} \frac{1}{2}\,\mathrm{erf}\left(\frac{\nu+|a|}{\sqrt{2\sigma^2}}\right) - \frac{1}{2}\,\mathrm{erf}\left(\frac{|a|-\nu}{\sqrt{2\sigma^2}}\right) & 0 \leqq \nu \leqq |a| \\ \frac{1}{2}\,\mathrm{erf}\left(\frac{\nu+|a|}{\sqrt{2\sigma^2}}\right) + \frac{1}{2}\,\mathrm{erf}\left(\frac{\nu+|a|}{\sqrt{2\sigma^2}}\right) & \nu > |a| \end{cases}$$

[Ref. 5, p. 136]

2. n = 2

$$F(\nu) = 1 - Q\left(\frac{a}{\sigma}, \frac{\nu}{\sigma}\right) \qquad \nu \geqq 0$$ [Ref. 19, p. 159]

3. n = 2k (even)

$$F(\nu) = 1 - Q_k\left(\frac{a}{\sigma}, \frac{\nu}{\sigma}\right) \qquad \nu \geqq 0$$

4. n = 2k+1 (odd)

— — — —

5. n

— — — —

II. RATIO

A. GAUSSIAN/GAUSSIAN

$$x \in N_1\left(a, \sigma_1^2\right), \qquad y \in N_1\left(b, \sigma_2^2\right)$$

Let

$$z = \frac{y}{z}, \qquad c = \frac{\sigma_2}{\sigma_1}$$

1. Independent

a. $a = b = 0$

$$F(z) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{z}{c}\right) \qquad \text{[Ref. 5, p. 30]}$$

b. $a \neq 0, \quad b = 0, \quad \sigma_1^2 = \sigma_2^2 = 1$

$$F(z) = \begin{cases} \frac{1}{\pi} \tan^{-1}\left(\frac{1}{|z|}\right) - 2V\left(\frac{a|z|}{\sqrt{1+z^2}}; \frac{a}{\sqrt{1+z^2}}\right) & z < 0 \\ \frac{1}{2} & z = 0 \\ 1 - \frac{1}{\pi} \tan^{-1}\left(\frac{1}{z}\right) + 2V\left(\frac{az}{\sqrt{1+z^2}}; \frac{a}{\sqrt{1+z^2}}\right) & z > 0 \end{cases}$$

[Ref. 29, p. 118]

c. $a \neq 0, \quad b \neq 0$

— — — —

2. Dependent

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

a. $a = b = 0$

$$F(z) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{\frac{z}{c} - \rho}{(1-\rho^2)^{1/2}}\right) \qquad \text{[Ref. 14]}$$

b. $a \neq 0, \quad b = 0$

— — — —

c. $a \neq 0, \quad b \neq 0$

— — — —

B. GAUSSIAN/RAYLEIGH (INDEPENDENT)

$$x \in N_1\left(0,\sigma_1^2\right), \qquad \underline{X} \in N_n\left(\underline{0},\sigma_2^2\right); \qquad r = \|\underline{X}\|$$

Let $\quad z = \dfrac{x}{r}, \qquad c = \dfrac{\sigma_2}{\sigma_1}$

1. n = 1

$$F(z) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} (cz) \qquad \text{[Ref. 5, p. 30]}$$

2. n = 2

$$F(z) = \frac{1}{2} + \frac{z}{2(c^2+z^2)^{1/2}} \qquad \text{[Ref. 5, p. 50]}$$

3. n = 3

$$F(z) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(cz) + \frac{cz}{\pi(1+c^2z^2)}$$

[Ref. 5, p. 30]

4. $n = 2k$ (even); $\sigma_1^2 = \sigma_2^2 = 1$

$$F(z) = \begin{cases} \frac{1}{2} - \frac{1}{2}\left(\frac{z^2}{1+z^2}\right)^{1/2} \sum_{j=1}^{k} \binom{2j-2}{j-1}\left(\frac{1}{4+4z^2}\right)^{j-1} & z < 0 \\ \frac{1}{2} + \frac{1}{2}\left(\frac{z^2}{1+z^2}\right)^{1/2} \sum_{j=1}^{k} \binom{2j-2}{j-1}\left(\frac{1}{4+4z^2}\right)^{j-1} & z \geq 0 \end{cases}$$

[Ref. 16, p. 345]

5. n

C. GAUSSIAN/RICE (INDEPENDENT)

$$x \in N_1\left(0,\sigma_1^2\right), \qquad \underline{X} \in N_n\left(\underline{A},\sigma_2^2\right)$$

$$a = \|\underline{A}\|, \qquad \nu = \|\underline{X}\|$$

Let $$q = \frac{x}{\nu}, \qquad c = \frac{\sigma_2}{\sigma_1}$$

1. $n = 1; \; \sigma_1^2 = \sigma_2^2 = 1$

$$F(q) = \begin{cases} \frac{1}{\pi} \tan^{-1}\left(\frac{1}{|q|}\right) - 2V\left(\frac{a|q|}{\sqrt{1+q^2}}, \frac{a}{\sqrt{1+q^2}}\right) & q < 0 \\ \frac{1}{2} & q = 0 \\ 1 - \frac{1}{\pi} \tan^{-1}\left(\frac{1}{q}\right) + 2V\left(\frac{aq}{\sqrt{1+q^2}}, \frac{a}{\sqrt{1+q^2}}\right) & q > 0 \end{cases}$$

[Ref. 29, p. 118]

2. $n = 2$

$$F(q) = \begin{cases} Q(\beta,\alpha) - \left(\frac{\sigma_2\alpha}{a}\right) \exp\left(-\frac{\alpha^2+\beta^2}{2}\right) I_0(\alpha\beta) & q < 0 \\ 1 - Q(\alpha,\beta) + \left(\frac{\sigma_2\beta}{a}\right) \exp\left(-\frac{\alpha^2+\beta^2}{2}\right) I_0(\alpha\beta) & q \geqq 0 \end{cases}$$

where

$$\alpha = \frac{a}{2\sigma_2}\left[1 - \frac{cq}{(1+c^2q^2)^{1/2}}\right]; \qquad \beta = \frac{a}{2\sigma_2}\left[1 + \frac{cq}{(1+c^2q^2)^{1/2}}\right]$$

[Ref. 29, p. 119]

3. $n = 2k$ (even); $\sigma_1^2 = \sigma_2^2 = 1$

$$F(q) = \begin{cases} Q(\alpha,\beta) - \left(\frac{\beta}{a}\right) \exp\left(-\frac{\alpha^2+\beta^2}{2}\right) I_0(\alpha\beta) - \frac{1}{2}\left(\frac{q^2}{1+q^2}\right)^{1/2} \\ \quad \cdot \sum_{j=2}^{k} \binom{2j-2}{j-1} \left(\frac{1}{4+4q^2}\right)^{j-1} {}_1F_1\left(\frac{1}{2}, j, -\frac{a^2}{2(1+q^2)}\right) & q < 0 \\ 1 - Q(\alpha,\beta) + \left(\frac{\beta}{a}\right) \exp\left(-\frac{\alpha^2+\beta^2}{2}\right) I_0(\alpha\beta) + \frac{1}{2}\left(\frac{q^2}{1+q^2}\right)^{1/2} \\ \quad \cdot \sum_{j=2}^{k} \binom{2j-2}{j-1} \left(\frac{1}{4+4q^2}\right)^{j-1} {}_1F_1\left(\frac{1}{2}, j, -\frac{a^2}{2(1+q^2)}\right) & q \geq 0 \end{cases}$$

where

$$\alpha = \frac{a}{2}\left[1 - \left(\frac{q^2}{1+q^2}\right)^{1/2}\right], \qquad \beta = \frac{a}{2}\left[1 + \left(\frac{q^2}{1+q^2}\right)^{1/2}\right]$$

[Ref. 16, p. 343]

4. n

D. RAYLEIGH/RAYLEIGH

1. Independent

$$\underline{X}^{(1)} \in N_n\left(\underline{0}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{0}, \sigma_2^2\right)$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let $$z = \frac{r_2}{r_1}, \qquad c = \frac{\sigma_2}{\sigma_1}$$

a. $n = 1, \quad m = 1$

$$F(z) = \frac{2}{\pi} \tan^{-1}\left(\frac{z}{c}\right) \qquad z \geqq 0 \qquad \text{[Ref. 5, p. 30]}$$

b. $n = 1, \quad m = 2$

$$F(z) = 1 - \left(\frac{c^2}{c^2 + z^2}\right)^{1/2} \qquad z \geqq 0 \qquad \text{[Ref. 5, p. 50]}$$

c. $n = 1, \quad m = 2k$ (even); $\sigma_1^2 = \sigma_2^2 = 1$

$$F(z) = 1 - \left(\frac{1}{1 + z^2}\right) \sum_{j=1}^{k} \binom{2j-2}{j-1} \left(\frac{z^2}{4 + 4z^2}\right)^{j-1} \qquad z \geqq 0$$

[Ref. 16, p. 343]

d. $n = 2, \quad m = 2$

$$F(z) = 1 - \frac{c^2}{1 + z^2} \qquad z \geqq 0 \qquad \text{[Ref. 5, p. 31]}$$

e. $n = m; \quad \sigma_1^2 = \sigma_2^2 = 1$ (Recursive Form)

$$F_n(z) = F_{n-2}(z) - \frac{h\,\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,(n-2)\,\Gamma\left(\frac{n-2}{2}\right)\,(1+h^2)^{(n-1)/2}} \qquad z \geqq 0$$

where

$$h = \frac{1 - z^2}{2z} \qquad \text{[Ref. 14]}$$

f. n even, m even

$$F(z) = 1 - \frac{1}{B\left(\frac{n}{2}, \frac{m}{2}\right)} \sum_{j=0}^{(m/2)-1} \binom{(m/2)-1}{j} \frac{(-1)^j}{\left(\frac{n}{2} + j\right)} \left(\frac{c^2}{c^2 + z^2}\right)^{(n/2)+j} \qquad z \geqq 0$$

[Ref. 29, p. 112]

g. n odd, m even; $\sigma_1^2 = \sigma_2^2 = 1$

$$F(z) = 1 - \left(\frac{1}{1+z^2}\right)^{1/2} - \left(\frac{1}{1+z^2}\right)^{n/2} \left\{ \sum_{k=0}^{(m-4)/2} \left(\frac{z^2}{1+z^2}\right)^{(m/2)-k-1} \cdot \frac{\Gamma\left(\frac{m+n}{2} - k - 1\right)}{\left(\frac{m}{2} - k - 1\right)!\;\Gamma\left(\frac{n}{2}\right)} - z^2 \sum_{k=0}^{(n-3)/2} (1+z^2)^k \right\} \qquad z \geqq 0$$

[Ref. 29, p. 119]

h. n odd, m odd

See Ref. 29, p. 118.

i. n, m

See Ref. 29.

2. Dependent

$$\underline{X}^{(1)} \in N_n(\underline{0},1), \qquad \underline{X}^{(2)} \in N_n(\underline{0},1)$$

$$\underline{M} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \qquad r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let

$$z = \frac{r_2}{r_1}$$

a. n = 1

$$F(z) = \frac{1}{2} - \frac{1}{\pi} \tan^{-1}\left(\frac{h}{\sqrt{1-\rho^2}}\right) \qquad z \geq 0$$

where

$$h = \frac{1 - z^2}{2z}$$

b. n = 2

$$F(z) = \frac{1}{2} - \frac{h}{2\sqrt{1-\rho^2+h^2}} \qquad z \geq 0$$

where

$$h = \frac{1 - z^2}{2z}$$

c. $n = 2k$ (even) (Recursive Form)

$$F_{2k}(z) = F_{2k-2}(z) - \frac{h(1-\rho^2)^{k-1}\,\Gamma\left(k - \frac{1}{2}\right)}{\sqrt{\pi}\,(2k-2)(k-2)!\,(1-\rho^2+h^2)^{k-(1/2)}} \qquad z \geqq 0$$

where

$$h = \frac{1 - z^2}{2z^2}$$

d. n (Recursive Form)

$$F_n(z) = F_{n-2}(z) - \frac{h(1-\rho^2)^{(n-2)/2}\,\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,(n-2)\,\Gamma\left(\frac{n-2}{2}\right)\,(1-\rho^2+h^2)^{(n-1)/2}} \qquad z \geqq 0$$

where

$$h = \frac{1 - z^2}{2z} \qquad \text{[Ref. 14]}$$

E. RICE/RAYLEIGH (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n\left(\underline{C}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{B}, \sigma_2^2\right)$$

$$r = \|\underline{X}^{(1)}\|, \qquad \nu = \|\underline{X}^{(2)}\|, \qquad b = \|\underline{B}\|$$

Let

$$u = \frac{\nu}{r}, \qquad c = \frac{\sigma_2}{\sigma_1}$$

1. $n = 1, \quad m = 1; \quad \sigma_1^2 = \sigma_2^2 = 1$

$$F(u) = \frac{2}{\pi} \tan^{-1} u - 4V\left(\frac{b}{\sqrt{1+u^2}}, \frac{bu}{\sqrt{1+u^2}}\right) \qquad u \geqq 0$$

[Ref. 29, p. 118]

2. $n = 1, \quad m = 2$

$$F(u) = 2\left[Q(\alpha,\beta) - \left(\frac{\sigma_2\beta}{b}\right) \exp\left(-\frac{\alpha^2 + \beta^2}{2}\right) I_0(\alpha\beta)\right] \qquad u \geqq 0$$

where

$$\alpha = \frac{b}{2\sigma_2}\left[1 - \left(\frac{c^2}{c^2 + u^2}\right)^{1/2}\right], \qquad \beta = \frac{b}{2\sigma_2}\left[1 + \left(\frac{c^2}{c^2 + u^2}\right)^{1/2}\right]$$

[Ref. 29, p. 119]

3. $n = 1, \quad m = 2k \quad (\text{even}); \quad \sigma_1^2 = \sigma_2^2 = 1$

$$F(u) = 2\left[Q(\alpha,\beta) - \left(\frac{\beta}{b}\right) \exp\left(-\frac{\alpha^2 + \beta^2}{2}\right) I_0(\alpha\beta)\right] - \left(\frac{1}{1 + u^2}\right)^{1/2} \exp\left(-\frac{b^2}{2(1+u^2)}\right)$$

$$\cdot \sum_{j=2}^{k} \binom{2j-2}{j-1} \left(\frac{u^2}{1 + u^2}\right)^{j-1} {}_1F_1\left(\frac{1}{2};\ j;\ -\frac{b^2u^2}{2(1+u^2)}\right) \qquad u \geqq 0$$

where

$$\alpha = \frac{b}{2}\left[1 - \left(\frac{1}{1 + u^2}\right)^{1/2}\right], \qquad \beta = \frac{b}{2}\left[1 + \left(\frac{1}{1 + u^2}\right)^{1/2}\right]$$

[Ref. 16, p. 343]

4. n = 2, m = 2

$$F(u) = \frac{\sigma_1^2 u^2}{\sigma_2^2 + \sigma_1^2 u^2} \exp\left(-\frac{b^2}{2\left(\sigma_2^2+\sigma_1^2 u^2\right)}\right) \qquad u \geqq 0$$

[Ref. 29, p. 110 and Appendix]

5. n = m

See Ref. 29.

6. n even, m even; $\sigma_1^2 = \sigma_2^2 = 1$

$$F(u) = \left(\frac{u^2}{1 + u^2}\right)^{m/2} \exp\left(-\frac{b^2}{2(1+u^2)}\right)$$

$$\cdot \sum_{j=0}^{(n/2)-1} \sum_{\ell=j}^{(n/2)-1} \frac{1}{j!\ 2^j} \binom{(m/2)+\ell-1}{\ell-j} \left(\frac{b^2u^2}{1+u^2}\right)^j \left(\frac{1}{1+u^2}\right)^\ell \qquad u \geqq 0$$

[Ref. 29, p. 112]

7. n odd, m even; $\sigma_1^2 = \sigma_2^2 = 1$

$$F(u) = 2\left[Q(\alpha,\beta) - \frac{\beta}{2} \exp\left(-\frac{\alpha^2+\beta^2}{2}\right) I_0(\alpha\beta)\right] - \left(\frac{1}{1+u^2}\right)^{n/2} \exp\left(-\frac{b^2}{2}\right)$$

$$\cdot \left\{\sum_{j=0}^{(m-4)/2} \left(\frac{u^2}{1+u^2}\right)^{(m-2j-2)/2} \frac{\Gamma\left(\frac{n+m-2j-2}{2}\right)}{\left(\frac{m-2j-2}{2}\right)!\ \Gamma\left(\frac{n}{2}\right)}\right.$$

$$\cdot\ {}_1F_1\left(\frac{n+m-2j-2}{2}; \frac{m-2j}{2}; 2\alpha\beta\right)$$

$$- u^{n-3} \sum_{j=0}^{(n-3)/2} (1 + u^2)^j \; {}_1F_1\left(\frac{n-2j}{2}; 1; 2\alpha\beta\right)\Bigg\} \qquad u \geq 0$$

where

$$\alpha = \frac{b}{2}\left[1 - \left(\frac{1}{1+u^2}\right)^{1/2}\right], \qquad \beta = \frac{b}{2}\left[1 + \left(\frac{1}{1+u^2}\right)^{1/2}\right]$$

[Ref. 29, p. 119]

8. $\underline{n, \; m}$

See Ref. 29.

F. RICE/RICE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n\left(\underline{A}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{B}, \sigma_2^2\right)$$

$$a = \|\underline{A}\|, \qquad b = \|\underline{B}\|, \qquad \nu_1 = \|\underline{X}^{(1)}\|, \qquad \nu_2 = \|\underline{X}^{(2)}\|$$

Let

$$q = \frac{\nu_2}{\nu_1}$$

1. $\underline{n = 1, \; m = 1; \; \sigma_1^2 = \sigma_2^2 = 1}$

$$F(q) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{q^2 - 1}{2q^2}\right) + 2V\left(\frac{aq+b}{\sqrt{1+q^2}}, \frac{a-bq}{\sqrt{1+q^2}}\right)$$

$$+ 2V\left(\frac{aq-b}{\sqrt{1+q^2}}, \frac{a+bq}{\sqrt{1+q^2}}\right) \qquad q \geq 0$$

[Ref. 29, p. 115]

2. $\underline{n = 2, \quad m = 2}$

$$F(q) = Q(\alpha,\beta) - \frac{\sigma_2^2\alpha^2}{a^2} \exp\left(-\frac{\alpha^2 + \beta^2}{2}\right) I_0(\alpha\beta) \qquad q \geq 0$$

where

$$\alpha = \left(\frac{a^2q^2}{\sigma_2^2 + \sigma_1^2q^2}\right)^{1/2}, \qquad \beta = \left(\frac{b^2}{\sigma_2^2 + \sigma_1^2q^2}\right)^{1/2}$$

[Ref. 29, p. 110]

For a generalization to dependent vectors see Ref. 35.

3. $\underline{n = m = 2k; \quad \sigma_1^2 = \sigma_2^2 = 1}$

$$F(q) = Q\left(\left(\frac{a^2q^2}{1 + q^2}\right)^{1/2}, \left(\frac{b^2}{1 + q^2}\right)^{1/2}\right) - \frac{1}{1 + q^2} \exp\left(-\frac{a^2q^2 + b^2}{2(1+q^2)}\right) I_0\left(\frac{abq}{1 + q^2}\right)$$

$$+ \exp\left(-\frac{a^2q^2 + b^2}{2(1 + q^2)}\right) \sum_{j=1-k}^{k-1} C_j(k-1,k-1;\ q) \left(\frac{bq}{a}\right)^j I_j\left(\frac{abq}{1 + q^2}\right) \qquad q \geq 0$$

where

$$C_j(k-1,k-1;\ q) = \begin{cases} \displaystyle\sum_{\ell=j}^{k-1} \binom{k+\ell-1}{\ell-j} \left(\frac{q^2}{1+q^2}\right)^k \left(\frac{1}{1+q^2}\right)^\ell - \delta_{0m}\left(\frac{q^2}{1+q^2}\right) & j \geq 0 \\ \\ -C_{-j}\left(k-1,k-1;\ \frac{1}{q}\right) & j < 0 \end{cases}$$

[Ref. 29, p. 112]

Note special case:

$$F(1) = Q\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right) - \frac{1}{2}\exp\left(-\frac{a^2+b^2}{4}\right) I_0\left(\frac{ab}{2}\right)$$

$$+ \exp\left(-\frac{a^2+b^2}{2}\right) \sum_{j=1}^{k-1} D_j \left[\left(\frac{b}{a}\right)^j - \left(\frac{a}{b}\right)^j\right] I_j\left(\frac{ab}{2}\right)$$

where

$$D_j = \sum_{\ell=j}^{k-1} \frac{(k-1+\ell)!\ 2^{-\ell-k}}{(k-1+j)!\ (\ell-j)!}$$

[Ref. 28, p. 17]

4. $n = m$

See Ref. 32.

5. n even, m even; $\sigma_1^2 = \sigma_2^2 = 1$

$$F(q) = Q\left(\left(\frac{a^2q^2}{1+q^2}\right)^{1/2}, \left(\frac{b^2}{1+q^2}\right)^{1/2}\right) - \frac{1}{1+q^2}\exp\left(-\frac{a^2q^2+b^2}{2(1+q^2)}\right) I_0\left(\frac{abq}{1+q^2}\right)$$

$$+ \exp\left(-\frac{a^2q^2+b^2}{2(1+q^2)}\right) \sum_{j=1-(m/2)}^{(n/2)-1} C_j\left(\frac{n}{2}-1, \frac{m}{2}-1; q\right)$$

$$\cdot \left(\frac{bq}{a}\right)^j I_j\left(\frac{abq}{1+q^2}\right) \qquad q \geq 0$$

where

$$C_j\left(\frac{n}{2}-1,\ \frac{m}{2}-1;\ q\right) = \begin{cases} \displaystyle\sum_{\ell=j}^{(n-2)/2} \binom{(m/2)+\ell-1}{\ell-j}\left(\frac{q^2}{1+q^2}\right)^{n/2}\left(\frac{1}{1+q^2}\right)^{\ell} - \delta_{m0}\left(\frac{q^2}{1+q^2}\right) & j \geqq 0 \\ -C_{-j}\left(\frac{m}{2}-1,\ \frac{n}{2}-1;\ \frac{1}{q}\right) & j < 0 \end{cases}$$

[Ref. 29, p. 112]

6. <u>n, m</u>

See Ref. 29.

III. SUM

A. CENTRAL CHI SQUARE (+) CENTRAL CHI SQUARE

1. Independent $\sigma_1^2 \neq \sigma_2^2$

$$\underline{X}^{(1)} \in N_n\left(\underline{0}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{0}, \sigma_2^2\right)$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let $$s = r_1^2 + r_2^2, \qquad c = \frac{\sigma_2}{\sigma_1}$$

a. $n = 1, \quad m = 1$

$$G(s) = \frac{2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2} \; \Lambda\left(\left|\frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right| \; ; \; \frac{s\left(\sigma_1^2 + \sigma_2^2\right)}{4\sigma_1^2\sigma_2^2}\right) \qquad s \geqq 0$$

b. $n = 2, \quad m = 2$

$$G(s) = \frac{1}{2|\sigma_1^2 - \sigma_2^2|}\left\{\frac{1}{\alpha}\left(1 - e^{-\alpha s}\right) - \frac{1}{\beta}\left(1 - e^{-\beta s}\right)\right\} \qquad s \geqq 0$$

where

$$\alpha = \frac{\sigma_1^2 + \sigma_2^2 - |\sigma_1^2 - \sigma_2^2|}{4\sigma_1^2\sigma_2^2}, \qquad \beta = \frac{\sigma_1^2 + \sigma_2^2 + |\sigma_1^2 - \sigma_2^2|}{4\sigma_1^2\sigma_2^2}$$

c. $n = m = 2k$

$$G(s) = \frac{1}{(k-1)!}\left(\frac{1}{2\left|\sigma_1^2 - \sigma_2^2\right|}\right)^k \sum_{j=0}^{k-1} \frac{(-1)^j (k+j-1)!}{j!} \left(\frac{2\sigma_1^2\sigma_2^2}{\left|\sigma_1^2 - \sigma_2^2\right|}\right)^j$$

$$\cdot \left\{\left[\frac{1}{\alpha^{k-j}} - e^{-\alpha s} \sum_{\ell=0}^{k-j-1} \frac{s^{k-j-\ell-1}}{(k-j-\ell-1)!\ \alpha^{\ell+1}}\right]\right.$$

$$\left. + (-1)^{k-j}\left[\frac{1}{\beta^{k-j}} - e^{-\beta s} \sum_{\ell=0}^{k-j-1} \frac{s^{k-j-\ell-1}}{(k-j-\ell-1)!\ \beta^{\ell+1}}\right]\right\} \qquad s \geqq 0$$

where

$$\alpha = \frac{\sigma_1^2 + \sigma_2^2 - \left|\sigma_1^2 - \sigma_2^2\right|}{4\sigma_1^2\sigma_2^2}, \qquad \beta = \frac{\sigma_1^2 + \sigma_2^2 + \left|\sigma_1^2 - \sigma_2^2\right|}{4\sigma_1^2\sigma_2^2}$$

[Ref. 5, p. 134]

d. $n = m$

— — — —

e. $n = 2, \quad m = 2k$

$$G(s) = 1 - \left(\frac{\sigma_1^2}{\sigma_1^2 - \sigma_2^2}\right)^k \left\{\exp\left(-\frac{s}{2\sigma_1^2}\right) + \exp\left(-\frac{s}{2\sigma_2^2}\right)\left(\frac{\sigma_2^2}{\sigma_1^2}\right) \sum_{j=0}^{k-1} \sum_{\ell=0}^{j} \frac{1}{(j-\ell)!}\right.$$

$$\left. \cdot \left(\frac{\sigma_1^2 - \sigma_2^2}{2\sigma_1^2\sigma_2^2}\right)^j \left(2\sigma_2^2\right)^\ell s^{j-\ell}\right\} \qquad s \geqq 0$$

[Ref. 5, p. 134]

f. n, m

— — — —

2. Dependent

$$\underline{X}^{(1)} \in N_n\left(\underline{0}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_n\left(\underline{0}, \sigma_2^2\right)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}; \qquad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let $$s = r_1^2 + r_2^2$$

a. $n = 1$

$$G(s) = \frac{2|\underline{W}|^{1/2}}{w_{11} + w_{22}} \Lambda\left(\frac{\left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{1/2}}{w_{11} + w_{22}}, \frac{(w_{11} + w_{22})s}{4}\right) \qquad s \geq 0$$

b. $n = 2$

$$G(s) = \frac{|\underline{W}|^{1/2}}{2\left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{1/2}} \left\{\frac{1}{\alpha}\left(1 - e^{-\alpha s}\right) - \frac{1}{\beta}\left(1 - e^{-\beta s}\right)\right\} \qquad s \geq 0$$

where

$$\alpha = \frac{1}{4}\left\{w_{11} + w_{22} - \left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{1/2}\right\}$$

$$\beta = \frac{1}{4}\left\{w_{11} + w_{22} + \left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{1/2}\right\}$$

c. $n = 2k$ (even)

$$G(s) = \frac{|\underline{W}|^k}{2^k (k-1)!\ \gamma^k} \sum_{j=0}^{k-1} \frac{(-1)^j (k+j-1)!}{j!} \left(\frac{2}{\gamma}\right)^j$$

$$\cdot \left\{ \left[\frac{1}{\alpha^{k-j}} - e^{-\alpha s} \sum_{\ell=0}^{k-j-1} \frac{s^{k-j-\ell-1}}{(k-j-\ell-1)!\ \alpha^{\ell+1}} \right] + (-1)^{k-j} \left[\frac{1}{\beta^{k-j}} - e^{-\beta s} \sum_{\ell=0}^{k-j-1} \frac{s^{k-j-\ell-1}}{(k-j-\ell-1)!\ \beta^{\ell+1}} \right] \right\} \qquad s \geq 0$$

where

$$\gamma = \left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{1/2}$$

$$\alpha = \frac{1}{4}\{w_{11} + w_{22} - \gamma\}$$

$$\beta = \frac{1}{4}\{w_{11} + w_{22} + \gamma\}$$

[Ref. 5, p. 134]

d. n

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B. CENTRAL CHI SQUARE (+) NON CENTRAL CHI SQUARE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n\left(\underline{A}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{0}, \sigma_2^2\right)$$

$$a = \|\underline{A}\|, \qquad \nu = \|\underline{X}^{(1)}\|, \qquad r = \|\underline{X}^{(2)}\|$$

Let $$w = \nu^2 + r^2$$

1. n = m

$$G(w) = \left(\frac{\sigma_1}{\sigma_2}\right)^n \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{n}{2} + j\right)}{j!\ \Gamma\left(\frac{n}{2}\right)} \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_2^2}\right)^j \left[1 - Q_{n+j}\left(\frac{a}{\sigma_1}, \frac{\sqrt{w}}{\sigma_1}\right)\right] \qquad w \geqq 0$$

2. n even, m even

$$G(w) = \left(\frac{\sigma_1}{\sigma_2}\right)^m \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{m}{2} + j\right)}{j!\ \Gamma\left(\frac{m}{2}\right)} \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_2^2}\right)^j \left[1 - Q_{(m+n)/2+j}\left(\frac{a}{\sigma_1}, \frac{\sqrt{w}}{\sigma_1}\right)\right] \qquad w \geqq 0$$

3. n, m

- - - -

C. NON CENTRAL CHI-SQUARE (+) NON CENTRAL CHI SQUARE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n\left(\underline{A}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{B}, \sigma_2^2\right)$$

$$a = \|\underline{A}\|, \qquad b = \|\underline{B}\|, \qquad \nu_1 = \|\underline{X}^{(1)}\|, \qquad \nu_2 = \|\underline{X}^{(2)}\|$$

Let $$t = \nu_1^2 + \nu_2^2$$

1. n = m

$$G(t) = \left(\frac{\sigma_1}{\sigma_2}\right)^n \exp\left(-\frac{b^2}{2\sigma_2^2}\right) \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{\Gamma\left(\frac{n}{2} + \ell + j\right)}{\ell!\ \Gamma\left(\frac{n}{2} + j\right) j!}$$

$$\cdot \left(\frac{b^2\sigma_1^2}{2\sigma_2^2}\right)^j \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_1^2}\right)^\ell \left[1 - Q_{n+j+\ell}\left(\frac{a}{\sigma_1}, \frac{\sqrt{t}}{\sigma_1}\right)\right] \qquad t \geq 0$$

2. n even, m even

$$G(t) = \left(\frac{\sigma_1}{\sigma_2}\right)^m \exp\left(-\frac{b^2}{2\sigma_1^2}\right) \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{\Gamma\left(\frac{m}{2} + j + \ell\right)}{j!\ \ell!\ \Gamma\left(\frac{m}{2} + j\right)} \left(\frac{b^2\sigma_1^2}{2\sigma_2^2}\right)^j \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_1^2}\right)^\ell$$

$$\cdot \left[1 - Q_{(m+n)/2+j+\ell}\left(\frac{a}{\sigma_1}, \frac{\sqrt{t}}{\sigma_1}\right)\right] \qquad t \geq 0$$

3. n, m

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IV. DIFFERENCE

A. CENTRAL CHI SQUARE (-) CENTRAL CHI SQUARE

1. Independent

$$\underline{X}^{(1)} \in N_n\left(\underline{0}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{0}, \sigma_2^2\right)$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let $$s = r_1^2 - r_2^2$$

a. $n = 1, \quad m = 1$

- - - -

b. $n = 2, \quad m = 2$

$$G(s) = \begin{cases} \dfrac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \exp\left(- \dfrac{s}{2\sigma_2^2}\right) & s < 0 \\[2ex] 1 - \dfrac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \exp\left(- \dfrac{s}{2\sigma_1^2}\right) & s \geqq 0 \end{cases}$$

c. $n = m = 2k$

$$G(s) = \begin{cases} a_k \exp\left(\dfrac{s}{2\sigma_2^2}\right) \displaystyle\sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j} \frac{(k+j-1)!}{j!\,(k-j-\ell-1)!} \left(\frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^j \\ \qquad \cdot \left(2\sigma_2^2\right)^{\ell+1} (-s)^{k-j-\ell} & s < 0 \\ 1 - a_k \exp\left(-\dfrac{s}{2\sigma_1^2}\right) \displaystyle\sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j} \frac{(k+j-1)!}{j!\,(k-j-\ell-1)!} \left(\frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^j \\ \qquad \cdot \left(2\sigma_1^2\right)^{\ell+1} s^{k-j-\ell} & s \geq 0 \end{cases}$$

where

$$a_k = \frac{1}{(k-1)!}\left(\frac{1}{2\left(\sigma_1^2+\sigma_2^2\right)}\right)^{k+1}$$

[Ref. 5, p. 134]

d. $n = m$

— — — —

e. $n = 2k, \quad m = 2$

$$G(s) = \begin{cases} \left(\dfrac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)^k \exp\left(\dfrac{s}{2\sigma_2^2}\right) & s < 0 \\[2ex] \left(\dfrac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)^k \left\{1 - 2\sigma_2^2 + 2\sigma_2^2\left(\dfrac{\sigma_1^2 + \sigma_2^2}{\sigma_2^2}\right)^k \left[1 - Q_k\left(0, \dfrac{\sqrt{s}}{\sigma_1}\right)\right]\right. & \\ \qquad \left. + 2\sigma_2^2 \exp\left(\dfrac{s}{2\sigma_2^2}\right) Q_k\left(0, \left(\dfrac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\sigma_2^2}\, s\right)^{1/2}\right)\right\} & s \geqq 0 \end{cases}$$

where

$$Q_k(0,b) = \exp\left(-\frac{b^2}{2}\right) \sum_{j=0}^{k-1} \frac{1}{j!}\left(\frac{b^2}{2}\right)^j$$

[Ref. 5, p. 134]

f. n even, m even; $\sigma_1^2 = \sigma_2^2 = 1$

$$G(s) = \begin{cases} \dfrac{1}{2^{n/2}} \exp\left(\dfrac{s}{2}\right) \displaystyle\sum_{k=0}^{(m-2)/2} \sum_{j=0}^{m/2-k-1} \dfrac{\left(\frac{n}{2} + k - i\right)}{2^k k!\left(\frac{n}{2} - 1\right)!\ j!} \left(\dfrac{-s}{2}\right)^j & s < 0 \\[2ex] ---- & s \geqq 0 \end{cases}$$

[Ref. 13]

g. n, m

- - - -

2. Dependent

$$\underline{X}^{(1)} \in N_n\left(\underline{0},\sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_n\left(\underline{0},\sigma_2^2\right)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \qquad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let $$s = r_1^2 - r_2^2$$

a. n = 1

- - - -

b. n = 2

$$G(s) = \begin{cases} \dfrac{2|\underline{W}|}{\gamma\alpha} \exp\left(\dfrac{\alpha}{4} s\right) & s < 0 \\ 1 - \dfrac{2|\underline{W}|}{\gamma\beta} \exp\left(-\dfrac{\beta}{4} s\right) & s \geqq 0 \end{cases}$$

where

$$\gamma = \left[(w_{11} + w_{22})^2 - 4w_{12}^2\right]^{1/2}$$

$$\alpha = \gamma - (w_{11} - w_{22}), \qquad \beta = \gamma + (w_{11} - w_{22})$$

c. $n = 2k$ (even)

$$G(s) = \begin{cases} a_k \exp\left(\frac{\alpha}{4} s\right) \sum_{j=0}^{k-1} \sum_{\ell=0}^{k+j-1} \frac{(k+j-1)!}{j!\,(k-j-\ell-1)!} \left(\frac{2}{\gamma}\right)^j \left(\frac{4}{\alpha}\right)^{\ell+1} (-s)^{k-j-\ell-1} & s < 0 \\ 1 - a_k \exp\left(-\frac{\beta}{4} s\right) \sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j-1} \frac{(k+j-1)!}{j!\,(k-j-\ell-1)!} \left(\frac{2}{\gamma}\right)^j \left(\frac{4}{\beta}\right)^{\ell+1} s^{k-j-\ell-1} & s \geqq 0 \end{cases}$$

where

$$a_k = \frac{|\underline{W}|^k}{2^k (k-1)!\ \gamma^k}$$

$$\gamma = \left[(w_{11} + w_{22})^2 - 4w_{12}^2\right]^{1/2}$$

$$\alpha = \gamma - (w_{11} - w_{22}), \qquad \beta = \gamma + (w_{11} - w_{22})$$

[Ref. 5, p. 134]

d. n

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B. NON CENTRAL CHI SQUARE (-) CENTRAL CHI SQUARE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n\left(\underline{A}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{0}, \sigma_2^2\right)$$

$$a = \|\underline{A}\|, \qquad \nu = \|\underline{X}^{(1)}\|, \qquad r = \|\underline{X}^{(2)}\|$$

Let $$w = \nu^2 - r^2$$

1. $n = 1, \quad m = 1$

— — — —

2. $n = 2, \quad m = 2$

$$G(w) = \begin{cases} d \exp\left(\dfrac{w}{2\sigma_2^2}\right) & w < 0 \\[2ex] d\left\{1 - 2\sigma_2^2 + 2\left(\sigma_1^2 + \sigma_2^2\right) \exp\left(\dfrac{a^2}{2\left(\sigma_1^2 + \sigma_2^2\right)}\right)\left[1 - Q\left(\dfrac{a}{\sigma_1}, \dfrac{\sqrt{w}}{\sigma_1}\right)\right]\right. & \\[2ex] \left. \quad + 2\sigma_2^2 \exp\left(\dfrac{w}{2\sigma_2^2}\right) Q\left(\left(\dfrac{a^2\sigma_2^2}{\sigma_1^2\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{1/2}, \left(\dfrac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\sigma_2^2} w\right)^{1/2}\right)\right\} & w \geqq 0 \end{cases}$$

where

$$d = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \exp\left(- \frac{a^2\left(\sigma_2^2 + 2\sigma_1^2\right)}{4\sigma_1^2\left(\sigma_1^2 + \sigma_2^2\right)}\right)$$

[Ref. 23]

3. $n = m$

— — — —

4. $n = 2, \quad m = 2k; \quad \sigma_1^2 = \sigma_2^2 = 1$

$$G(w) = \begin{cases} \dfrac{1}{2} \exp\left(\dfrac{2w - a^2}{4}\right) \displaystyle\sum_{j=0}^{k-1} \dfrac{1}{2^j} L_j\left(-\dfrac{a^2}{2}\right) \sum_{\ell=0}^{k-j-1} \dfrac{1}{\ell!}\left(-\dfrac{w}{2}\right)^{\ell} & w < 0 \\ ---- & w \geq 0 \end{cases}$$

[Ref. 13]

5. $n = 2k, \quad m = 2$

$$G(w) = \begin{cases} d \exp\left(\dfrac{w}{2\sigma_2^2}\right) & w < 0 \\ d\left\{1 - 2\sigma_2^2 + 2\sigma_2^2\left(\dfrac{\sigma_1^2 + \sigma_2^2}{\sigma_2^2}\right)^k \exp\left(\dfrac{a^2}{2\left(\sigma_1^2 + \sigma_2^2\right)}\right)\left[1 - Q_k\left(\dfrac{a}{\sigma_1}, \dfrac{\sqrt{w}}{\sigma_1}\right)\right]\right. \\ \left. \quad + 2\sigma_2^2 \exp\left(\dfrac{w}{2\sigma_2^2}\right) Q_k\left(\left(\dfrac{a^2\sigma_2^2}{2\sigma_1^2\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{1/2}, \left(\dfrac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\sigma_2^2}\, w\right)^{1/2}\right)\right\} & w \geq 0 \end{cases}$$

where

$$d = \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)^k \exp\left(-\frac{a^2\left(\sigma_2^2 + 2\sigma_1^2\right)}{4\sigma_1^2\left(\sigma_1^2 + \sigma_2^2\right)}\right)$$

6. <u>n even, m even; $\sigma_1^2 = \sigma_2^2 = 1$</u>

$$G(w) = \begin{cases} \dfrac{1}{2^{n/2}} \exp\left(\dfrac{2w - a^2}{4}\right) \displaystyle\sum_{k=0}^{(m/2)-1} \dfrac{1}{2^k} L_k^{(n/2)-1}\left(-\dfrac{a^2}{2}\right) \\ \qquad \cdot \displaystyle\sum_{j=0}^{(m/2)-k-1} \dfrac{1}{j!}\left(-\dfrac{w}{2}\right)^j & w < 0 \\ ---- & w \geqq 0 \end{cases}$$

[Ref. 13]

7. <u>n, m</u>

C. NON CENTRAL CHI SQUARE (-) NON CENTRAL CHI SQUARE

V. PRODUCT

A. GAUSSIAN WITH ZERO MEAN

1. Independent

$$\underline{X}^{(1)} \in N_n\left(\underline{0}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_n\left(\underline{0}, \sigma_2^2\right)$$

Let $$z = \underline{X}^{(1)'}\,\underline{X}^{(2)} = \left(\underline{X}^{(1)}, \underline{X}^{(2)}\right)$$

a. $n = 1$

----

b. $n = 2, \quad c = \sigma_1 \sigma_2$

$$F(z) = \begin{cases} \frac{1}{2} \exp\left(\frac{z}{c}\right) & z < 0 \\ 1 - \frac{1}{2} \exp\left(-\frac{z}{c}\right) & z \geq 0 \end{cases}$$

c. $n = 2k, \quad c = \sigma_1 \sigma_2$

$$F(z) = \begin{cases} \dfrac{\exp\left(\frac{z}{c}\right)}{2^{k-2} c^k (k-1)!} \displaystyle\sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j-1} \frac{(k+j-1)!}{2^j j!\,(k-j-\ell-1)!}\, c^{j+\ell+1} (-z)^{k-j-\ell-1} & z < 0 \\ 1 - \dfrac{\exp\left(-\frac{z}{c}\right)}{2^{k-2} c^k (k-1)!} \displaystyle\sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j-1} \frac{(k+j-1)!}{2^j j!\,(k-j-\ell-1)!}\, c^{j+\ell+1} z^{k-j-\ell-1} & z \geq 0 \end{cases}$$

[Ref. 5, p. 134]

d. n

- - - -

e. Angle

- - - -

2. Dependent

$$\underline{X}^{(1)} \in N_n\left(\underline{0},\sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_n\left(\underline{0},\sigma_2^2\right)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \qquad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

Let $$z = \left(\underline{X}^{(1)}, \underline{X}^{(2)}\right)$$

a. n = 1

- - - -

b. n = 2

$$F(z) = \begin{cases} \dfrac{1-\rho}{2} \exp\left(\dfrac{z}{\sigma_1\sigma_2(1-\rho)}\right) & z < 0 \\[2ex] 1 - \dfrac{1+\rho}{2} \exp\left(-\dfrac{z}{\sigma_1\sigma_2(1+\rho)}\right) & z \geqq 0 \end{cases}$$

c. $n = 2k, \quad c = \sigma_1\sigma_2, \quad d = \sqrt{w_{11}w_{22}}$

$$F(z) = \begin{cases} \dfrac{e^{\alpha z}}{2^k(k-1)!c^k} \displaystyle\sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j-1} \frac{(k+j-1)!}{j!\,(k-j-\ell-1)!} \left(\frac{1}{2d}\right)^j \cdot \left(\frac{1}{\alpha}\right)^{\ell+1} (-z)^{k-j-\ell-1} & z < 0 \\[2ex] 1 - \dfrac{e^{-\beta z}}{2^k(k-1)!c^k} \displaystyle\sum_{j=0}^{k-1} \sum_{\ell=0}^{k-j-1} \frac{(k+j-1)!}{j!\,(k-j-\ell-1)!} \left(\frac{1}{2d}\right)^j \cdot \left(\frac{1}{\beta}\right)^{\ell+1} z^{k-j-\ell-1} & z \geqq 0 \end{cases}$$

where

$$\alpha = d - w_{12} = \sqrt{w_{11}w_{22}} - w_{12}$$

$$\beta = d + w_{12} = \sqrt{w_{11}w_{22}} + w_{12}$$

[Ref. 5, p. 134]

d. n

- - - -

e. Angle

- - - -

B. GAUSSIAN WITH ONE NON ZERO MEAN (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{A},\sigma^2), \qquad \underline{X}^{(2)} \in N_n(\underline{0},\sigma^2), \qquad a = \|\underline{A}\|$$

Let $$z = \left(\underline{X}^{(1)}, \underline{X}^{(2)}\right)$$

1. n = 1

----

2. n = 2

$$F(z) = \begin{cases} \frac{1}{2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \exp\left(\frac{z}{\sigma^2}\right) \sum_{j=0}^{\infty} \sum_{\ell=0}^{j} \sum_{r=0}^{j-\ell} C_{j\ell r} \frac{a^{2j} (-z)^{j-\ell-r}}{2^{2j+\ell}(\sigma^2)^{2j-\ell-r}} & z < 0 \\ 1 - \frac{1}{2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \exp\left(-\frac{z}{\sigma^2}\right) \sum_{j=0}^{\infty} \sum_{\ell=0}^{j} \sum_{r=0}^{j-\ell} C_{j\ell r} \frac{a^{2j} z^{j-\ell-r}}{2^{2j+\ell}(\sigma^2)^{2j-\ell-r}} & z \geq 0 \end{cases}$$

where

$$C_{j\ell r} = \frac{(j + \ell)!}{j!\ j!\ \ell!\ (j-\ell-r)!}$$

3. n = 2k (even)

$$F(z) = \begin{cases} \left(\frac{1}{2\sigma^2}\right)^k \exp\left(-\frac{a^2}{2\sigma^2}\right) \exp\left(\frac{z}{\sigma^2}\right) \sum_{j=0}^{\infty} \sum_{\ell=0}^{k+j-1} \sum_{r=0}^{k+j-\ell-1} d_{j\ell r} \cdot \frac{a^{2j} (-z)^{k+j-\ell-r-1}}{2^{\ell+2j}(\sigma^2)^{2j-\ell-r-1}} & z < 0 \\ 1 - \left(\frac{1}{2\sigma^2}\right)^k \exp\left(-\frac{a^2}{2\sigma^2}\right) \exp\left(-\frac{z}{\sigma^2}\right) \sum_{j=0}^{\infty} \sum_{\ell=0}^{k+j-1} \sum_{r=0}^{k+j-\ell-1} d_{j\ell r} \cdot \frac{a^{2j} z^{k+j-\ell-r-1}}{2^{\ell+2j}(\sigma^2)^{2j-\ell-r-1}} & z \geq 0 \end{cases}$$

where

$$d_{j\ell r} = \frac{(k+j+\ell-1)!}{j!\ (k+j-1)!\ \ell!\ (k+j-\ell-r-1)!}$$

[Ref. 5, p. 134]

4. $\underline{n}$

– – – –

C. GAUSSIAN WITH NON ZERO MEANS

– – – –

D. RAYLEIGH $(\times)$ RAYLEIGH

1. Independent

$$\underline{X}^{(1)} \in N_n\left(\underline{0},\sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{0},\sigma_2^2\right)$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let $\quad z = r_1 r_2, \qquad c = \sigma_1\sigma_2$

a. $n = 1, \quad m = 1$

– – – –

b. $n = 2, \quad m = 2$

$$F(z) = 1 - \frac{z}{c} K_1\left(\frac{z}{c}\right) \qquad z \geq 0 \qquad \text{[Ref. 6]}$$

c. $n = m = 2k$

– – – –

d. $n, \quad m$

– – – –

2. Dependent

- - - -

E. RAYLEIGH (×) RICE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n\left(\underline{A}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{0}, \sigma_2^2\right)$$

$$a = \|\underline{A}\|, \qquad \nu = \|\underline{X}^{(1)}\|, \qquad r = \|\underline{X}^{(2)}\|$$

Let $$u = \nu r$$

1. $n = 1, \quad m = 1$

- - - -

2. $n = 2, \quad m = 2$

$$F(u) = 1 - \exp\left(-\frac{a^2}{2\sigma_1^2}\right) \sum_{j=0}^{\infty} \left(\frac{1}{j!}\right)^2 \left(\frac{a}{2\sigma_1}\right)^{2j} \left(\frac{u}{\sigma_1\sigma_2}\right)^{j+1} k_{j+1}\left(\frac{u}{\sigma_1\sigma_2}\right) \qquad u \geqq 0$$

3. $n, \quad m$

- - - -

F. RICE (×) RICE

- - - -

I. FUNDAMENTAL VARIABLES

A. GAUSSIAN

$$x \in N_1(a,\sigma^2)$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-a)^2}{2\sigma^2}\right)$$

$$\psi_x(\mu) = \exp\left(i\mu a - \frac{1}{2}\mu^2\sigma^2\right)$$

For $a = 0$

$$Ex^{2k} = \sum_{j=0}^{[(k-1)/2]} \left(\frac{k!}{j!\ 2^j}\right)^2 \frac{\sigma^{2k}}{(k-2j)!} \qquad k \geqq 0$$

B. RAYLEIGH

$$\underline{X} \in N_n(\underline{0},\sigma^2), \qquad r = \|\underline{X}\|, \qquad s = r^2$$

1. $\underline{n = 1}$

$$f(r) = \frac{2}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{r^2}{2\sigma^2}\right) \qquad r \geqq 0$$

$$Er^k = \frac{(2\sigma^2)^{k/2}}{\sqrt{\pi}} \Gamma\left(\frac{k+1}{2}\right) \qquad k \geqq 0$$

$$\psi_s(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2}\right)^{1/2}$$

2. n = 2

$$f(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) \qquad r \geq 0$$

$$Er^k = (2\sigma^2)^{k/2} \Gamma\left(\frac{k}{2} + 1\right) \qquad k \geq 0$$

$$\psi_s(\mu) = \frac{1}{1 - 2i\mu\sigma^2}$$

3. n = 2k (even)

$$f(r) = \frac{2r^{2k-1}}{(2\sigma^2)^k (k-1)!} \exp\left(-\frac{r^2}{2\sigma^2}\right) \qquad r \geq 0$$

$$Er^\ell = (2\sigma^2)^{\ell/2} \frac{\Gamma\left(k + \frac{\ell}{2}\right)}{(k-1)!} \qquad \ell \geq 0$$

$$\psi_s(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2}\right)^k$$

4. n

$$f(r) = \frac{2r^{n-1}}{(2\sigma^2)^{n/2} \Gamma\left(\frac{n}{2}\right)} \exp\left(-\frac{r^2}{2\sigma^2}\right) \qquad r \geq 0$$

[Ref. 24. p. 29]

$$Er^k = (2\sigma^2)^{k/2} \frac{\Gamma\left(\frac{n + k}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \qquad k \geq 0$$

[Ref 24, p. 72]

$$\psi_s(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2}\right)^{n/2}$$

C. RICE

$$\underline{X} \in N_n(\underline{A},\sigma^2), \qquad a = \|\underline{A}\|, \qquad \nu = \|\underline{X}\|, \qquad t = \nu^2$$

1. n = 1

$$f(\nu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\nu^2 + a^2}{2\sigma^2}\right)\left[\exp\left(\frac{\nu a}{\sigma^2}\right) + \exp\left(-\frac{\nu a}{\sigma^2}\right)\right] \qquad \nu \geqq 0$$

$$E\nu^k = (2\sigma^2)^{k/2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}}\ {}_1F_1\left(\frac{k+1}{2}, \frac{1}{2}; \frac{a^2}{2\sigma^2}\right) \qquad k \geqq 0$$

$$\psi_t(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2}\right)^{1/2} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma^2}\right)$$

2. n = 2

$$f(\nu) = \frac{\nu}{\sigma^2} \exp\left(-\frac{\nu^2 + a^2}{2\sigma^2}\right) I_0\left(\frac{\nu a}{\sigma^2}\right) \qquad \nu \geqq 0$$

$$E\nu^k = (2\sigma^2)^{k/2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \Gamma\left(\frac{1}{2}k+1\right)\ {}_1F_1\left(\frac{1}{2}k+1, 1; \frac{a^2}{2\sigma^2}\right)$$

$$\psi_t(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2}\right) \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma^2}\right)$$

3. n = 2k (even)

$$f(\nu) = \frac{\nu^k}{\sigma^2 a^{k-1}} \exp\left(-\frac{\nu^2 + a^2}{2\sigma^2}\right) I_{k-1}\left(\frac{\nu a}{\sigma^2}\right) \qquad \nu \geq 0$$

$$E\nu^\ell = (2\sigma^2)^{\ell/2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \frac{\Gamma\left(k + \frac{\ell}{2}\right)}{(k-1)!} \; {}_1F_1\left(k + \frac{\ell}{2}; k; \frac{a^2}{2\sigma^2}\right) \qquad \ell \geq 0$$

$$\Psi_t(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2}\right)^k \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma^2}\right)$$

4. n = 2k + 1 (odd)

$$f(\nu) = \frac{1}{\sqrt{2\pi\sigma^2}} \left(\frac{\nu}{a}\right)^k \exp\left(-\frac{\nu^2 + a^2}{2\sigma^2}\right) \left\{ \exp\left(\frac{\nu a}{\sigma^2}\right) \sum_{j=0}^{k-1} \frac{(-1)^j \, (k+j-1)!}{j! \, (k-j-1)!} \left(\frac{\sigma^2}{2\nu a}\right)^j \right.$$

$$\left. + (-1)^k \exp\left(-\frac{\nu a}{\sigma^2}\right) \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j! \, (k-j-1)!} \left(\frac{\sigma^2}{2\nu a}\right)^j \right\} \qquad \nu \geq 0$$

$$E\nu^\ell = (2\sigma^2)^{\ell/2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \frac{\Gamma\left(k + \frac{\ell+1}{2}\right)}{\Gamma\left(k + \frac{1}{2}\right)} \; {}_1F_1\left(k + \frac{\ell+1}{2}, k + \frac{1}{2}; \frac{a^2}{2\sigma^2}\right)$$

$$\Psi_t(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2}\right)^{k+(1/2)} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma^2}\right)$$

5. n

$$f(\nu) = \frac{a}{\sigma^2}\left(\frac{\nu}{a}\right)^{n/2} \exp\left(-\frac{\nu^2 + a^2}{2\sigma^2}\right) I_{(n-2)/2}\left(\frac{\nu a}{\sigma^2}\right) \qquad \nu \geq 0$$

[Ref. 24, p. 28]

$$E\nu^k = (2\sigma^2)^{k/2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \frac{\Gamma\left(\frac{n+k}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \; {}_1F_1\left(\frac{n+k}{2}, \frac{n}{2}; \frac{a^2}{2\sigma^2}\right) \qquad k \geq 0.$$

[Ref. 24, p. 72]

$$\psi_t(\mu) = \left(\frac{1}{1 - 2i\mu\sigma^2}\right)^{n/2} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma^2}\right)$$

D. JOINT DENSITIES

1. Gaussian

a. Let

$$\underline{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

be a gaussian vector with $E\underline{Y} = \underline{A}$ and $\underline{K} = E(\underline{Y} - \underline{A})'(\underline{Y} - A)$. Then

$$f(\underline{Y}) = \frac{1}{(2\pi)^{n/2} \, |\underline{K}|^{1/2}} \exp\left(-\frac{1}{2}(\underline{Y} - \underline{A})' \, \underline{K}^{-1}(\underline{Y} - \underline{A})\right)$$

b. Let

$$\underline{U} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}$$

Then,

$$\psi_{\underline{z}}(\underline{U}) = \exp\left(i\underline{U}'\underline{A} - \frac{1}{2}\,\underline{U}'\underline{K}\;\underline{U}\right)$$

c. For $n = 2$, $\underline{A} = 0$,

$$\underline{K} = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}$$

$$Ey_1^{k_1}\, y_2^{k_2} = \begin{cases} 0 & k_1 + k_2 \text{ odd} \\ \left(\frac{1}{2}\right)^{(k_2-k_1)/2} \sigma^{(k_1+k_2)}\, k_1!\, k_2! \\ \quad \cdot \displaystyle\sum_{j=0}^{k_1} \frac{(2j)!\, \rho^{k_1-2j}}{(j!\, 2^j)^2\, (k_1-2j)!\, \left(2j + \frac{k_2 - k_1}{2}\right)!} & k_1 + k_2 \text{ even},\ k_2 \geq k_1 \end{cases}$$

2. <u>Rayleigh</u>

$$\underline{X}^{(1)} \in N_n\left(\underline{0},\sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_n\left(\underline{0},\sigma_2^2\right)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}; \qquad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

Let $\qquad r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$

$$f(r_1,r_2) = \frac{(r_1r_2)^{n/2}}{\left(2|w_{12}|\right)^{(n-2)/2} \quad |\underline{M}|^{n/2}\,\Gamma\left(\frac{n}{2}\right)} \exp\left(-\frac{1}{2}\left(w_{11}r_1^2 + w_{22}r_2^2\right)\right)$$

$$\cdot\, I_{(n-2)/2}\left(r_1r_2|w_{12}|\right) \qquad r_1,r_2 \geqq 0$$

[Ref. 24, p. 34]

$$Er_1^{k_1}\,r_2^{k_2} = \frac{2^{(k_1+k_2)/2}|\underline{W}|^{n/2}\,\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2}{2}\right)}{w_{11}^{(n+k_1)/2}\;w_{22}^{(n+k_2)/2}\;\Gamma^2\left(\frac{n}{2}\right)}$$

$$\cdot\, {}_2F_1\left(\frac{n+k_1}{2}, \frac{n+k_2}{2}, \frac{n}{2}; \frac{w_{12}^2}{w_{11}w_{22}}\right) \qquad k_1,k_2 > 0$$

[Ref. 24, p. 73]

3. <u>Rice</u>

$$\underline{X}^{(1)} \in N_n\left(\underline{A},\sigma_1^2\right), \qquad \underline{X}^{(2)} \in N\left(\underline{A},\sigma_2^2\right)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} ; \qquad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

Let $\quad a = \|\underline{A}\|, \qquad \nu_1 = \|\underline{X}^{(1)}\|, \qquad \nu_2 = \|\underline{X}^{(2)}\|$

$$f(\nu_1, \nu_2) = \frac{\nu_1 \nu_2 \, \Gamma\left(\frac{n-2}{2}\right)}{|\underline{M}|^{n/2}} \left(\frac{2}{a^2 w_1 w_{12} w_2}\right)^{(1/2)(n-2)}$$

$$\cdot \exp\left(- \frac{1}{2}\left(w_{11}\nu_1^2 + w_{22}\nu_2^2\right) - \frac{1}{2}\,(w_1 + w_2)a^2\right)$$

$$\cdot \sum_{j=0}^{\infty} (-1)^j \left(\frac{1}{2}\,n + j - 1\right) \binom{n+j-3}{n-3} I_{(1/2)n+j-1}(\nu_1 \nu_2 w_{12})$$

$$\cdot \; I_{(1/2)n+j-1}(\nu_1 a w_1) \; I_{(1/2)n+j-1}(\nu_2 a w_2)$$

where

$$w_1 = w_{11} + w_{12}, \qquad w_2 = w_{22} + w_{12}$$

[Ref. 24, p. 32]

II. RATIO

A. GAUSSIAN/GAUSSIAN

$$x \in N_1\left(a, \sigma_1^2\right), \qquad y \in N_1\left(b, \sigma_2^2\right), \qquad M = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

Let $$z = \frac{y}{x}, \qquad c = \frac{\sigma_2}{\sigma_1}$$

1. Independent

$$\rho = 0$$

a. $a = b = 0$

$$f(z) = \frac{c}{\pi(c^2 + z^2)}$$

$$\psi_z(\mu) = e^{-|\mu|c}$$

b. $a \neq 0, \quad b = 0$

$$f(z) = \frac{\sigma_1\sigma_2}{\pi\left(\sigma_2^2 + \sigma_1^2 z^2\right)} \exp\left(-\frac{a^2}{2\sigma_1^2}\right) + \frac{a\sigma_2^2}{\sqrt{2\pi}\left(\sigma_2^2 + \sigma_1^2 z^2\right)^{3/2}}$$

$$\cdot \exp\left(-\frac{a^2 z^2}{2\left(\sigma_2^2 + \sigma_1^2 z^2\right)}\right) \operatorname{erf}\left(\frac{a\sigma_2^2}{\sqrt{2}\,\sigma_1\sigma_2\left(\sigma_2^2 + \sigma_1^2 z^2\right)^{1/2}}\right)$$

[Ref. 4, p. 60]

c. $a \neq 0, \quad b \neq 0$

$$f(z) = \frac{\sigma_1\sigma_2}{\pi\left(\sigma_2^2 + \sigma_1^2 z^2\right)} \exp\left(-\frac{1}{2}\left(\frac{b^2}{\sigma_2^2} + \frac{a^2}{\sigma_1^2}\right)\right) + \frac{a\sigma_2^2 + b\sigma_1^2 z}{\sqrt{2\pi}\left(\sigma_2^2 + \sigma_1^2 z^2\right)^{3/2}}$$

$$\cdot \exp\left(-\frac{(b - az)^2}{2\left(\sigma_2^2 + \sigma_1^2 z^2\right)}\right) \operatorname{erf}\left(\frac{a\sigma_2^2 + b\sigma_1^2 z}{\sqrt{2}\,\sigma_1\sigma_2\left(\sigma_2^2 + \sigma_1^2 z^2\right)^{1/2}}\right)$$

[Ref. 4, p. 60]

2. Dependent

a. $a = b = 0$

$$f(z) = \frac{c(1 - \rho^2)^{1/2}}{\pi(c^2 - 2\rho c z + z^2)}$$

$$\psi_z(\mu) = \exp\left(i\mu\rho c - |\mu|(1 - \rho^2)^{1/2}\, c\right)$$

b. $a \neq 0, \quad b = 0$

$$f(z) = \frac{\sigma_1\sigma_2(1 - \rho^2)^{1/2}}{\pi\left(\sigma_2^2 - 2\rho z\sigma_1\sigma_2 + \sigma_1^2 z^2\right)} \exp\left(-\frac{a^2}{2\sigma_1^2(1 - \rho^2)}\right) + \frac{a\rho\sigma_1 z - a\sigma_2^2}{\sqrt{2\pi}\left(\sigma_2^2 - 2\rho\sigma_1\sigma_2 z + \sigma_1^2 z^2\right)^{3/2}}$$

$$\cdot \exp - \frac{a^2 z^2}{2\left(\sigma_2^2 - 2\rho\sigma_1\sigma_2 z + \sigma_1^2 z^2\right)}$$

$$\cdot \operatorname{erf}\left(\frac{a\sigma_1\sigma_2\rho z - a\sigma_2^2}{\sqrt{2}\,\sigma_1\sigma_2(1 - \rho^2)\left(\sigma_2^2 - 2\sigma_1\sigma_2\rho z + \sigma_1^2 z^2\right)^{1/2}}\right)$$

c. $a \neq 0, \quad b \neq 0$

$$f(z) = \frac{\sigma_1\sigma_2(1-\rho^2)^{1/2}}{\pi\left(\sigma_2^2 - 2\sigma_1\sigma_2\rho z + \sigma_1^2 z^2\right)} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{b^2}{\sigma_2^2} - \frac{2\rho ab}{\sigma_1\sigma_2} + \frac{a^2}{\sigma_1^2}\right)\right)$$

$$+ \frac{\sigma_2(b\sigma_1\rho - a\sigma_2) + z\sigma_1(a\rho - b\sigma_1)}{\sqrt{2\pi}\left(\sigma_2^2 - 2\rho\sigma_1\sigma_2 z + \sigma_1^2 z^2\right)^{3/2}} \exp\left(-\frac{1}{2}\left(\frac{(b - za)^2}{\sigma_2^2 - 2\sigma_1\sigma_2\rho + \sigma_1^2 z^2}\right)\right)$$

$$\cdot \operatorname{erf}\left(\frac{\sigma_2(\rho b\sigma_1 - a\sigma_2) + z\sigma_1(\rho a\sigma_2 - b\sigma_1)}{\sqrt{2}\,\sigma_1\sigma_2(1-\rho^2)\left(\sigma_2^2 - 2\rho\sigma_1\sigma_2 z + \sigma_1^2 z^2\right)^{1/2}}\right)$$

[Ref. 4, p. 60]

B. GAUSSIAN/RAYLEIGH (INDEPENDENT)

$$x \in N_1\left(0,\sigma_1^2\right); \qquad \underline{X} \in N_n\left(\underline{0},\sigma_2^2\right); \qquad r = \|\underline{X}\|$$

Let

$$z = \frac{x}{r}, \qquad c = \frac{\sigma_2}{\sigma_1}$$

1. <u>n = 1</u>

$$f(z) = \frac{c}{\pi(1 + c^2 z^2)}$$

$$\psi_z(\mu) = e^{-|\mu|/c}$$

2. <u>n = 2</u>

$$f(z) = \frac{c}{2(1 + c^2 z^2)^{3/2}}$$

3. <u>n = 3</u>

$$f(z) = \frac{2c}{\pi(1 + c^2z^2)^2}$$

$$\psi_z(\mu) = \left(1 + \frac{|\mu|}{c}\right) e^{-|\mu|/c}$$

4. <u>n = 2k (even)</u>

$$f(z) = \frac{c\Gamma\left(k + \frac{1}{2}\right)}{\sqrt{\pi}\,(k-1)!\,(1 + c^2z^2)^{k+(1/2)}}$$

5. <u>n</u>

$$f(z) = \frac{c\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)(1 + c^2z^2)^{(n+1)/2}} = \frac{c}{B\left(\frac{1}{2}n, \frac{1}{2}\right)(1 + c^2z^2)^{(n+1)/2}}$$

[Ref. 24, p. 57]

C. GAUSSIAN/RICE (INDEPENDENT)

$$x \in N_1\left(0, \sigma_1^2\right), \qquad \underline{X} \in N_n\left(\underline{A}, \sigma_2^2\right), \qquad a = \|\underline{A}\|, \qquad \nu = \|\underline{X}\|$$

Let
$$q = \frac{x}{\nu}, \qquad c = \frac{\sigma_2}{\sigma_1}$$

1. <u>n = 1</u>

$$f(q) = \frac{\sigma_1\sigma_2}{\pi\left(\sigma_1^2 + \sigma_2^2q^2\right)} \exp\left(-\frac{a^2}{2\sigma_2^2}\right) {}_1F_1\left(1, \frac{1}{2}, \frac{a^2\sigma_1^2}{2\sigma_2^2\left(\sigma_1^2 + \sigma_2^2q^2\right)}\right)$$

2. n = 2

$$f(q) = \frac{\sigma_1^2\sigma_2}{2\left(\sigma_1^2 + \sigma_2^2 q^2\right)^{3/2}} \exp\left(-\frac{a^2}{4\sigma_2^2}\left(\frac{\sigma_1^2 + 2\sigma_2^2 q^2}{\sigma_1^2 + \sigma_2^2 q^2}\right)\right)$$

$$\cdot \left[\left(1 + \frac{a^2\sigma_1^2}{2\sigma_2^2\left(\sigma_1^2 + \sigma_2^2 q^2\right)}\right) I_0\left(\frac{a^2\sigma_1^2}{4\sigma_2^2\left(\sigma_1^2 + \sigma_2^2 q^2\right)}\right)\right.$$

$$\left. + \frac{a^2\sigma_1^2}{2\sigma_2^2\left(\sigma_1^2 + \sigma_2^2 q^2\right)} I_1\left(\frac{a^2\sigma_1^2}{4\sigma_2^2\left(\sigma_1^2 + \sigma_2^2 q^2\right)}\right)\right]$$

3. n = 2k (even)

$$f(q) = \frac{\sigma_2}{\sigma_1 B\left(k, \frac{1}{2}\right)\left(1 + \frac{\sigma_2^2}{\sigma_1^2} z^2\right)^{k+(1/2)}}$$

$$\cdot \exp\left(-\frac{a^2}{2\sigma_2^2}\right) {}_1F_1\left(k + \frac{1}{2}, k, \frac{a^2\sigma_1^2}{2\sigma_2^2\left(\sigma_1^2 + \sigma_2^2 q^2\right)}\right)$$

4. <u>n</u>

$$f(q) = \frac{\sigma_2}{\sigma_1 B\left(\frac{1}{2}n, \frac{1}{2}\right)\left(1 + \frac{\sigma_2^2}{\sigma_1^2} q^2\right)^{(n+1)/2}}$$

$$\cdot \exp\left(-\frac{a^2}{2\sigma_2^2}\right) {}_1F_1\left(\frac{n+1}{2}, \frac{n}{2}; \frac{a^2\sigma_1^2}{2\sigma_2^2\left(\sigma_1^2 + \sigma_2^2 q^2\right)}\right)$$

[Ref. 24, p. 56]

D. RAYLEIGH/RAYLEIGH

1. Independent

$$\underline{X}^{(1)} \in N_n\left(\underline{0}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{0}, \sigma_2^2\right)$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let $$z = \frac{r_2}{r_1}, \qquad c = \frac{\sigma_2}{\sigma_1}$$

a. $n = m = 1$

$$f(z) = \frac{2c}{\pi(c^2 + z^2)} \qquad z \geq 0$$

$$\psi_z(\mu) = e^{-|\mu|c}$$

b. $n = 1, \quad m = 2$

$$f(z) = \frac{zc^2}{(c^2 + z^2)^{3/2}} \qquad z \geqq 0$$

c. $n = 1, \quad m = 2k$

$$f(z) = \frac{2z^{2k-1} \; c^{2k}}{B\left(\frac{1}{2}, k\right) \; (c^2 + z^2)^{k+(1/2)}} \qquad z \geqq 0$$

d. $n = m = 2$

$$f(z) = \frac{2zc^2}{(c^2 + z^2)^2} \qquad z \geqq 0$$

$$Ez = \frac{\pi}{2c}$$ [Ref. 24, p. 73]

e. $n = m$

$$f(z) = \frac{2z^{n-1} \; c^n}{B\left(\frac{n}{2}, \frac{n}{2}\right) \; (c^2 + z^2)^n} \qquad z \geqq 0$$

$$Ez^k = \frac{\Gamma\left(\frac{n+k}{2}\right) \; \Gamma\left(\frac{n-k}{2}\right)}{c^k \left[\Gamma\left(\frac{n}{2}\right)\right]^{1/2}} \qquad 0 \leqq k \leqq n$$

f. n even, m even

$$f(z) = \frac{2z^{m-1} \; c^m}{B\left(\frac{n}{2}, \frac{m}{2}\right) \; (c^2 + z^2)^{(m+n)/2}} \qquad z \geqq 0$$

[Ref. 24, p. 52]

g. n odd, m even

Same as f.

h. n odd, m odd

Same as f.

i. n, m

Same as f.

2. Dependent

$$\underline{X}^{(1)} \in N_n\left(\underline{0},\sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_n\left(\underline{0},\sigma_2^2\right)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \qquad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let $$z = \frac{r_2}{r_1}$$

a. $n = 1$

$$f(z) = \frac{2|\underline{W}|^{1/2}\left(w_{11} + w_{22}z^2\right)}{\pi\left[\left(w_{11} + w_{22}z^2\right)^2 - 4w_{12}^2z^2\right]} \qquad z \geq 0$$

b. n = 2

$$f(z) = \frac{2|\underline{W}| \left(w_{11} + w_{22}z^2\right)}{\left[\left(w_{11} + w_{22}z^2\right)^2 - 4w_{12}^2 z^2\right]^{3/2}} \qquad z \geqq 0$$

$$Ez = \frac{|\underline{W}| \ \pi}{2w_{11}^{3/2} \ w_{22}} \ {}_2F_1\left(\frac{3}{2}, \frac{1}{2}, 1; \frac{w_{12}^2}{w_{11}w_{22}}\right)$$

c. n = 2k

$$f(z) = \frac{2|\underline{W}|^k \ z^{2k-1} \left(w_{11} + w_{22}z^2\right)}{B(k,k) \left[\left(w_{11} + w_{22}z^2\right)^2 - 4w_{12}^2 z^2\right]^{k+(1/2)}} \qquad z \geqq 0$$

$$Ez^{\ell} = \frac{|\underline{W}|^k \ \Gamma\left(k + \frac{\ell}{2}\right)\Gamma\left(k - \frac{\ell}{2}\right)}{w_{11}^{k+(\ell/2)} w_{22}^{k-(\ell/2)} [(k-1)!]^2} \ {}_2F_1\left(k + \frac{\ell}{2}, k - \frac{\ell}{2}, k; \frac{w_{12}^2}{w_{11}w_{22}}\right) \qquad 0 \leqq \ell < 2k$$

d. n

$$f(z) = \frac{2|\underline{W}|^{n/2} \ z^{n-1}\left(w_{11} + w_{22}z^2\right)}{B\left(\frac{1}{2}n, \frac{1}{2}n\right)\left[\left(w_{11} + w_{22}z^2\right)^2 - 4w_{12}^2 z^2\right]^{(n+1)/2}} \qquad z \geqq 0$$

[Ref. 24, p. 50]

$$Ez^k = \frac{|\underline{W}|^{n/2} \ \Gamma\left(\frac{n+k}{2}\right)\Gamma\left(\frac{n-k}{2}\right)}{w_{11}^{(n+k)/2} \ w_{22}^{(n-k)/2} \left[\Gamma\left(\frac{n}{2}\right)\right]^2} \ {}_2F_1\left(\frac{n+k}{2}, \frac{n-k}{2}, \frac{n}{2}; \frac{w_{12}^2}{w_{11}w_{22}}\right) \qquad 0 \leqq k < n$$

[Ref. 24, p. 73]

E. RICE/RAYLEIGH (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n\left(\underline{0},\sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{B},\sigma_2^2\right)$$

$$b = \|\underline{B}\|, \qquad r = \|\underline{X}^{(1)}\|, \qquad \nu = \|\underline{X}^{(2)}\|, \qquad c = \frac{\sigma_2}{\sigma_1}$$

Let
$$u = \frac{\nu}{r}$$

1. n = 1, m = 1

$$f(u) = \frac{2\sigma_1\sigma_2}{\pi\left(\sigma_2^2 + \sigma_1^2u^2\right)} \exp\left(-\frac{b^2}{2\sigma_2^2}\right) {}_1F_1\left(1, \frac{1}{2}; \frac{u^2b^2\sigma_1^2}{2\sigma_2^2\left(\sigma_2^2 + \sigma_1^2u^2\right)}\right) \qquad u \geqq 0$$

2. n = 1, m = 2

$$f(u) = \frac{u\sigma_1^2\sigma_2}{\left(\sigma_2^2 + \sigma_1^2u^2\right)^{3/2}} \exp\left(-\frac{\alpha^2+\beta^2}{2}\right) \{(1 + 2\alpha\beta)I_0(\alpha\beta) + 2\alpha\beta I_1(\alpha\beta)\} \qquad u \geqq 0$$

where

$$\alpha = \frac{b}{2\sigma_2}\left[1 - \left(\frac{\sigma_2^2}{\sigma_2^2 + \sigma_1^2u^2}\right)^{1/2}\right], \qquad \beta = \frac{b}{2\sigma_2}\left[1 + \left(\frac{\sigma_2^2}{\sigma_2^2 + \sigma_1^2u^2}\right)^{1/2}\right]$$

3. <u>n = 1, m = 2k (even)</u>

$$f(u) = \frac{2u^{2k-1}\sigma_1^{2k}\sigma_2}{B\left(\frac{1}{2},k\right)\left(\sigma_2^2+\sigma_1^2u^2\right)^{k+(1/2)}} \exp\left(-\frac{b^2}{2\sigma_2^2}\right) {}_1F_1\left(k+\frac{1}{2}, k; \frac{u^2b^2\sigma_1^2}{2\sigma_2^2\left(\sigma_2^2+\sigma_1^2u^2\right)}\right)$$

$$u \geqq 0$$

4. <u>n = 2, m = 2</u>

$$f(u) = \frac{2u\sigma_1^2\sigma_2^2}{\left(\sigma_2^2+\sigma_1^2u^2\right)^2} \exp\left(-\frac{b^2}{2\left(\sigma_2^2+\sigma_1^2u^2\right)}\right)\left\{1+\frac{u^2b^2\sigma_1^2}{2\sigma_2^2\left(\sigma_2^2+\sigma_1^2u^2\right)}\right\} \qquad u \geqq 0$$

5. <u>n = m</u>

$$f(u) = \frac{2u^{n-1}\sigma_1^n\sigma_2^n}{B\left(\frac{n}{2},\frac{n}{2}\right)\left(\sigma_2^2+\sigma_1^2u^2\right)^n} \exp\left(-\frac{b^2}{2\sigma_2^2}\right) {}_1F_1\left(n, \frac{n}{2}; \frac{u^2b^2\sigma_1^2}{2\sigma_2^2\left(\sigma_2^2+\sigma_1^2u^2\right)}\right) \qquad u \geqq 0$$

6. <u>n even, m even</u>

$$f(u) = \frac{2u^{m-1}\ \sigma_1^m\sigma_2^n}{B\left(\frac{n}{2},\frac{m}{2}\right)\ \left(\sigma_2^2+\sigma_1^2u^2\right)^{(m+n)/2}}$$

$$\cdot \exp\left(-\frac{b^2}{2\sigma_2^2}\right) {}_1F_1\left(\frac{m+n}{2}, \frac{m}{2}; \frac{u^2b^2\sigma_1^2}{2\sigma_2^2\left(\sigma_2^2+\sigma_1^2u^2\right)}\right) \qquad u \geqq 0$$

[Ref. 24, p. 52]

7. <u>n odd, m even</u>

Same as 6.

8. <u>n, m</u>

Same as 6.

F. RICE/RICE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n\left(\underline{A}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{B}, \sigma_2^2\right)$$

$$a = \|\underline{A}\|, \qquad b = \|\underline{B}\|, \qquad \nu_1 = \|\underline{X}^{(1)}\|, \qquad \nu_2 = \|\underline{X}^{(2)}\|$$

Let
$$q = \frac{\nu_2}{\nu_1}$$

1. <u>n = 1, m = 1</u>

$$f(q) = \frac{2\sigma_1\sigma_2}{\sigma_2^2 + \sigma_1^2 q^2} \exp\left(-\frac{1}{2}\left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2}\right)\right) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \binom{k+j}{j} \frac{1}{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)}$$

$$\cdot \left(\frac{a^2\sigma_2^2}{2\sigma_1^2}\right)^k \left(\frac{q^2 b^2 \sigma_1^2}{2\sigma_2^2}\right)^j \frac{1}{\left(\sigma_2^2 + \sigma_1^2 q^2\right)^{k+j}} \qquad q \geq 0$$

(or Differentiate Distribution)

2. n = 2, m = 2

$$f(q) = \frac{2q\sigma_1^2\sigma_2^2}{\left(\sigma_2^2 + \sigma_1^2 q^2\right)^2} \exp\left(-\frac{a^2q^2 + b^2}{2\left(\sigma_2^2 + \sigma_1^2q^2\right)}\right)$$

$$\cdot \left[\left(1 + \frac{a^2\sigma_2^4 + q^2b^2\sigma_1^4}{2\sigma_1^2\sigma_2^2\left(\sigma_2^2 + \sigma_1^2q^2\right)}\right) I_0\left(\frac{qab}{\sigma_2^2 + \sigma_1^2q^2}\right) + \frac{qab}{\sigma_2^2 + \sigma_1^2q^2}\right.$$

$$\left. \cdot\, I_1\left(\frac{qab}{\sigma_2^2 + \sigma_1^2q^2}\right)\right] \qquad q \geqq 0$$

3. n = m = 2k

$$f(q) = (-1)^k \frac{ab}{\sigma_1^2\sigma_2^2}\left(\frac{q}{ab}\right)^k \exp\left(-\frac{1}{2}\left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2}\right)\right)$$

$$\cdot \frac{d^k}{dh^k}\left[\frac{1}{2h}\exp\left(\frac{1}{4h}\left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2}q^2\right)\right) I_{k-1}\left(\frac{qab}{2h\sigma_1^2\sigma_2^2}\right)\right] \qquad q \geqq 0$$

evaluated at $h = \frac{1}{2}\left(\frac{1}{\sigma_1^2} + \frac{q^2}{\sigma_2^2}\right)$ [Ref. 24, p. 52]

4. <u>n = m</u>

$$f(q) = \frac{2q^{n-1}\,\sigma_1^n\sigma_2^n}{\left(\sigma_2^2 + \sigma_1^2 q^2\right)^n} \exp\left(-\frac{1}{2}\left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2}\right)\right) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(n+k+j-1)!}{k!\ j!\ \Gamma\left(\frac{n}{2} + k\right)\Gamma\left(\frac{n}{2} + j\right)}$$

$$\cdot \left(\frac{a^2\sigma_2^2}{2\sigma_1^2}\right)^k \left(\frac{q^2 b^2 \sigma_1^2}{2\sigma_2^2}\right)^j \frac{1}{\left(\sigma_2^2 + \sigma_1^2 q^2\right)^{k+j}} \qquad q \geqq 0$$

5. <u>n even, m even</u>

$$f(q) = \frac{2a^{m-1}\,\sigma_1^m\sigma_2^n}{\left(\sigma_2^2 + \sigma_1^2 q^2\right)^{(m+n)/2}} \exp\left(-\frac{1}{2}\left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2}\right)\right)$$

$$\cdot \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{m+n}{2} + k + j\right)}{k!\ j!\ \Gamma\left(\frac{n}{2} + k\right)\Gamma\left(\frac{m}{2} + j\right)} \left(\frac{a^2\sigma_2^2}{2\sigma_1^2}\right)^k \left(\frac{q^2 b^2 \sigma_1^2}{2\sigma_2^2}\right)^j$$

$$\cdot \frac{1}{\left(\sigma_2^2 + \sigma_1^2 q^2\right)^{k+j}} \qquad q \geqq 0$$

[Ref. 24, p. 51]

(or Differentiate Distribution)

6. <u>n, m</u>

Same as 5.

G. JOINT DENSITY

$$\underline{X}^{(1)} \in N_n\left(\underline{0}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_n\left(\underline{0}, \sigma_2^2\right)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \qquad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

Also

$$\underline{X}^{(3)} \in N_n\left(\underline{0}, \sigma_1^2\right), \qquad \underline{X}^{(4)} \in N_n\left(\underline{0}, \sigma_2^2\right), \qquad \underline{M}$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|, \qquad r_3 = \|\underline{X}^{(3)}\|, \qquad r_4 = \|\underline{X}^{(4)}\|$$

Let $z_1 = \dfrac{r_3}{r_1}$ and $z_2 = \dfrac{r_4}{r_2}$

Here r_1 and r_3 are independent

r_2 and r_4 are independent

r_1 and r_2 are related by $\underline{M}$

r_3 and r_4 are related by $\underline{M}$

$$f(z_1, z_2) = \frac{4|\underline{W}|^n (z_1 z_2)^{n-1}}{\Gamma^2\left(\frac{n}{2}\right)\left[w_{11}w_{22}(1+z_1)^2(1+z_2)^2\right]^n} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{[(n+k+j-1)!]^2}{k!\ j!\ \Gamma\left(\frac{n}{2}+k\right)\Gamma\left(\frac{n}{2}+j\right)}$$

$$\cdot \left(\frac{w_{12}^2}{w_{11}w_{22}\left(1+z_1^2\right)\left(1+z_2^2\right)}\right)^{k+j} (z_1 z_2)^{2j} \qquad z_1, z_2 \geqq 0$$

[Ref. 24, p. 54]

III. SUM

A. CENTRAL CHI SQUARE (+) CENTRAL CHI SQUARE

1. Independent $\sigma_1^2 \neq \sigma_2^2$

$$\underline{X}^{(1)} \in N_n\left(\underline{0}, \sigma_1^2\right); \qquad \underline{X}^{(2)} \in N_m\left(\underline{0}, \sigma_2^2\right)$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let $$s = r_1^2 + r_2^2, \qquad c = \frac{\sigma_2}{\sigma_1}$$

a. $n = m = 1$

$$g(s) = \frac{1}{2\sigma_1\sigma_2} \exp\left(- \frac{\left(\sigma_1^2 + \sigma_2^2\right)s}{4\sigma_1^2\sigma_2^2}\right) I_0\left(\frac{\left|\sigma_1^2 - \sigma_2^2\right| s}{4\sigma_1^2\sigma_2^2}\right) \qquad s \geq 0$$

$$Es^k = \frac{2^{2k+1}\, k!\, \sigma_1^{2k+1}\, \sigma_2^{2k+1}}{\left(\sigma_1^2 + \sigma_2^2\right)^{k+1}}\; {}_2F_1\left(\frac{k+1}{2}, \frac{k}{2} + 1, 1; \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)^2\right) \qquad k \geq 0$$

$$\psi_s(\mu) = \left(\frac{1}{\left(1 - 2i\mu\sigma_1^2\right)\left(1 - 2i\mu\sigma_2^2\right)}\right)^{1/2}$$

b. $n = m = 2$

$$g(s) = \frac{1}{2\left|\sigma_1^2 - \sigma_2^2\right|} \exp\left(- \frac{\left(\sigma_1^2 + \sigma_2^2\right)s}{4\sigma_1^2\sigma_2^2}\right)\left\{\exp\left(\frac{\left|\sigma_1^2 - \sigma_2^2\right| s}{4\sigma_1^2\sigma_2^2}\right) - \exp\left(- \frac{\left|\sigma_1^2 - \sigma_2^2\right| s}{4\sigma_1^2\sigma_2^2}\right)\right\}$$

$$s \geq 0$$

$$Es^{k} = \frac{2^{2k+2}\ (k+1)!\ \sigma_1^{2k+2}\ \sigma_2^{2k+2}}{\left(\sigma_1^2 + \sigma_2^2\right)^{k+2}}\ {}_2F_1\left(\frac{k}{2} + 1,\ \frac{k+3}{2}\ ,\ \frac{3}{2}\ ;\ \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)^2\right) \qquad k \geqq 0$$

$$\psi_s(\mu) = \frac{1}{\left(1 - 2i\mu\sigma_1^2\right)\ \left(1 - 2i\mu\sigma_2^2\right)}$$

c. $n = m = 2k$

$$g(s) = \frac{1}{s(k-1)!}\left(\frac{s}{2\left|\sigma_1^2 - \sigma_2^2\right|}\right)^{k} \exp\left(-\ \frac{\left(\sigma_1^2 + \sigma_2^2\right)s}{4\sigma_1^2\sigma_2^2}\right)\left\{\exp\left(\frac{s\left|\sigma_1^2 - \sigma_2^2\right|}{4\sigma_1^2\sigma_2^2}\right)\right.$$

$$\cdot \sum_{j=0}^{k-1} \frac{(-1)^j\ (k+j-1)!}{j!\ (k+j-1)!}\left(\frac{2\sigma_1^2\sigma_2^2}{\left(\sigma_1^2 - \sigma_2^2\right)s}\right)^{j} + (-1)^k \exp\left(-\ \frac{s\left|\sigma_1^2 - \sigma_2^2\right|}{4\sigma_1^2\sigma_2^2}\right)$$

$$\left.\cdot \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j!\ (k-j-1)!}\left(\frac{2\sigma_1^2\sigma_2^2}{s\left|\sigma_1^2 - \sigma_2^2\right|}\right)^{r}\right\} \qquad s \geqq 0$$

$$Es^{\ell} = \frac{2^{2k+2\ell}\ (2k+\ell-1)!\ \sigma_1^{2k+2\ell}\ \sigma_2^{2k+2\ell}}{(2k-1)!\ \left(\sigma_1^2 + \sigma_2^2\right)^{2k+\ell}}$$

$$\cdot\ {}_2F_1\left(k + \frac{\ell}{2},\ k + \frac{\ell+1}{2}\ ,\ k + \frac{1}{2}\ ;\ \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)^2\right) \qquad \ell \geqq 0$$

$$\psi_s(\mu) = \left(\frac{1}{\left(1 - 2i\mu\sigma_1^2\right)\ \left(1 - 2i\mu\sigma_2^2\right)}\right)^{k}$$

d. $n = m$

$$g(s) = \frac{\sqrt{\pi}}{2\sigma_1\sigma_2\Gamma\left(\frac{n}{2}\right)}\left(\frac{s}{2\left|\sigma_1^2 - \sigma_2^2\right|}\right)^{(n-1)/2} \exp\left(-\frac{\left(\sigma_1^2 + \sigma_2^2\right)s}{4\sigma_1^2\sigma_2^2}\right)$$

$$\cdot\ I_{(n-1)/2}\left(\frac{\left|\sigma_1^2 - \sigma_2^2\right|s}{4\sigma_1^2\sigma_2^2}\right) \qquad s \geq 0$$

$$Es^k = \frac{2^{n+2k}\ (n+k-1)!\ \sigma_1^{n+2k}\ \sigma_2^{n+2k}}{(n-1)!\ \left(\sigma_1^2 + \sigma_2^2\right)^{k+n}}$$

$$\cdot\ {}_2F_1\left(\frac{n+k}{2}, \frac{n+k+1}{2}, \frac{n}{2}; \left(\frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)^2\right) \qquad k \geq 0$$

[Ref. 24, p. 74]

$$\psi_s(\mu) = \left(\frac{1}{\left(1 - 2i\mu\sigma_1^2\right)\left(1 - 2i\mu\sigma_2^2\right)}\right)^{n/2}$$

e. $n = 2, \quad m = 2k$

$$g(s) = \frac{\sigma_2^{2k-2}}{2\left(\sigma_2^2 - \sigma_1^2\right)^k} \exp\left(-\frac{s}{2\sigma_1^2}\right)$$

$$\cdot \left\{\exp\left(\frac{s\left(\sigma_2^2 - \sigma_1^2\right)}{2\sigma_1^2\sigma_2^2}\right) - \sum_{j=0}^{k-1} \frac{1}{j!}\left(\frac{s\left(\sigma_2^2 - \sigma_1^2\right)}{2\sigma_1^2\sigma_2^2}\right)^j\right\} \qquad s \geq 0$$

$$\psi_s(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2}\right) \left(\frac{1}{1 - 2i\mu\sigma_2^2}\right)^k$$

f. n, m

$$g(s) = \frac{s^{(n+m-2)/2}}{2^{(n+m)/2}\, \sigma_1^n \sigma_2^m\, \Gamma\left(\frac{n+m}{2}\right)} \exp\left(-\frac{s}{2\sigma_1^2}\right) {}_1F_1\left(\frac{m}{2}, \frac{n+m}{2}, \frac{\left(\sigma_2^2 - \sigma_1^2\right)s}{2\sigma_1^2\sigma_2^2}\right) \qquad s \geqq 0$$

[Ref. 24, p. 61]

$$\psi_s(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2}\right)^{n/2} \left(\frac{1}{1 - 2i\mu\sigma_2^2}\right)^{m/2}$$

2. Dependent

$$\underline{X}^{(1)} \in N_n\left(\underline{0}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_n\left(\underline{0}, \sigma_2^2\right)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \qquad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let $$s = r_1^2 + r_2^2$$

a. n = 1

$$g(s) = \frac{\exp\left(-\frac{(w_{11} + w_{22})s}{4}\right)}{2|\underline{M}|^{1/2}} I_0\left(\frac{1}{4} s \left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{1/2}\right) \quad s \geqq 0$$

$$Es^k = \frac{2^{2k+1} k!}{|\underline{M}|^{1/2} (w_{11} + w_{22})^{k+1}} \; {}_2F_1\left(\frac{k+1}{2}, \frac{k}{2} + 1, 1; \frac{(w_{11} - w_{22})^2 + 4w_{12}^2}{(w_{11} + w_{22})^2}\right) \quad k \geqq 0$$

$$\psi_s(\mu) = \left(\frac{|\underline{M}|}{\left(\sigma_2^2 - 2i\mu|\underline{M}|\right)\left(\sigma_1^2 - 2i\mu|\underline{M}|\right) - \rho^2\sigma_1^2\sigma_2^2}\right)^{1/2}$$

b. n = 2

$$g(s) = \frac{1}{2|\underline{M}|\left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{1/2}} \{e^{-\alpha s} - e^{-\beta s}\} \quad s \geqq 0$$

$$\alpha = \frac{1}{4}\left\{w_{11} + w_{22} - \left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{1/2}\right\}$$

$$\beta = \frac{1}{4}\left\{w_{11} + w_{22} + \left[(w_{11} + w_{22})^2 + 4w_{12}^2\right]^{1/2}\right\}$$

$$Es^k = \frac{2^{2k+2}(k+1)!}{|\underline{M}|(w_{11} + w_{22})^{k+2}} \; {}_2F_1\left(\frac{k}{2} + 1, \frac{k+3}{2}, \frac{3}{2}; \frac{(w_{11} - w_{22})^2 + 4w_{12}^2}{(w_{11} + w_{22})^2}\right) \quad k \geqq 0$$

$$\psi_s(\mu) = \frac{|\underline{M}|}{\left(\sigma_2^2 - 2i\mu|\underline{M}|\right)\left(\sigma_1^2 - 2i\mu|\underline{M}|\right) - \rho^2\sigma_1^2\sigma_2^2}$$

c. $n = 2k$

$$g(s) = \frac{s^{k-1}}{(2|\underline{M}|)^k \, (k-1)! \, \gamma^k} \left\{ e^{-\alpha s} \sum_{j=0}^{k-1} \frac{(-1)^j \, (k+j-1)!}{j! \, (k-j-1)!} \left(\frac{2}{s\gamma}\right)^j \right.$$

$$\left. + (-1)^k \, e^{-\beta s} \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j! \, (k-j-1)!} \left(\frac{2}{s\gamma}\right)^j \right\} \qquad s \geqq 0$$

$$\gamma = \left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{1/2}$$

$$\alpha = \frac{1}{4} \left[w_{11} + w_{22} - \gamma\right]$$

$$\beta = \frac{1}{4} \left[w_{11} + w_{22} + \gamma\right]$$

$$Es^{\ell} = \frac{2^{2k+2\ell} \, (2k+\ell-1)!}{|\underline{M}|^k \, (2k-1)! \, (w_{11} + w_{22})^{2k+\ell}}$$

$$\cdot \, {}_2F_1\left(k + \frac{\ell}{2}, \, k + \frac{\ell+1}{2}, \, k + \frac{1}{2}; \, \frac{(w_{11} - w_{22})^2 + 4w_{12}^2}{(w_{11} + w_{22})^2}\right) \qquad \ell \geqq 0$$

$$\psi_s(\mu) = \left(\frac{|\underline{M}|}{\left(\sigma_2^2 - 2i\mu|\underline{M}|\right)\left(\sigma_1^2 - 2i\mu|\underline{M}|\right) - \rho^2\sigma_1^2\sigma_2^2}\right)^k$$

d. n

$$g(s) = \frac{\left(\frac{\pi}{2}\right)^{1/2} s^{(n-1)/2} \exp\left(- \frac{(w_{11} + w_{22})s}{4}\right)}{(2|\underline{M}|)^{n/2} \Gamma\left(\frac{n}{2}\right)\left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{(n-1)/2}}$$

$$\cdot I_{(n-1)/2}\left(\frac{1}{4} s \left[(w_{11} - w_{22})^2 + 4w_{12}^2\right]^{1/2}\right) \quad s \geq 0$$

[Ref. 24, p. 58]

$$Es^k = \frac{2^{n+2k} (n+k-1)!}{|\underline{M}|^{n/2} (n-1)! \ (w_{11} + w_{22})^{n+k}}$$

$$\cdot {}_2F_1\left(\frac{n+k}{2}, \frac{n+k+1}{2}, \frac{n+1}{2}; \frac{(w_{11} - w_{22})^2 + 4w_{12}^2}{(w_{11} + w_{22})^2}\right) \quad k \geq 0$$

[Ref. 24, p. 74]

$$\psi_s(\mu) = \left(\frac{|\underline{M}|}{\left(\sigma_2^2 - 2i\mu|\underline{M}|\right)\left(\sigma_1^2 - 2i\mu|\underline{M}|\right) - \rho^2\sigma_1^2\sigma_2^2}\right)^{n/2}$$

B. CENTRAL CHI SQUARE (+) NON CENTRAL CHI SQUARE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n\left(\underline{A}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{0}, \sigma_2^2\right)$$

$$a = \|\underline{A}\|, \qquad \nu = \|\underline{X}^{(1)}\|. \qquad r = \|\underline{X}^{(2)}\|$$

Let $\qquad w = \nu^2 + r^2$

1. n = m

$$g(w) = \frac{1}{2\sigma_1^2}\left(\frac{\sigma_1}{\sigma_2}\right)^n \left(\frac{w}{a^2}\right)^{(n-1)/2} \exp\left(-\frac{w + a^2}{2\sigma_1^2}\right)$$

$$\cdot \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{n}{2} + j\right)}{j!\ \Gamma\left(\frac{n}{2}\right)} \left(\frac{\sqrt{w}\left(\sigma_2^2 - \sigma_1^2\right)}{a\sigma_2^2}\right)^j I_{n+j-1}\left(\frac{\sqrt{w}\ a}{\sigma_1^2}\right) \qquad w \geqq 0$$

$$\psi_w(\mu) = \left(\frac{1}{\left(1 - 2i\mu\sigma_2^2\right)\left(1 - 2i\mu\sigma_1^2\right)}\right)^{n/2} \exp\left(-\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right)$$

2. n even, m even

$$g(w) = \frac{1}{2\sigma_1^2}\left(\frac{\sigma_1}{\sigma_2}\right)^m \left(\frac{w}{a^2}\right)^{(m+n-2)/4} \exp\left(-\frac{w + a^2}{2\sigma_1^2}\right)$$

$$\cdot \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{m}{2} + j\right)}{j!\ \Gamma\left(\frac{m}{2}\right)} \left(\frac{\sqrt{w}\left(\sigma_2^2 - \sigma_1^2\right)}{a\sigma_2^2}\right)^j I_{[(m+n)/2]+j-1}\left(\frac{\sqrt{w}\ a}{\sigma_1^2}\right)$$

[Ref. 24, p. 60]

$$\psi_w(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_2^2}\right)^{m/2} \left(\frac{1}{1 - 2i\mu\sigma_1^2}\right)^{n/2} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right)$$

3. n, m

Same as 2.

C. NON CENTRAL CHI SQUARE (+) NON CENTRAL CHI SQUARE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n\left(\underline{A}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{B}, \sigma_2^2\right)$$

$$a = \|\underline{A}\|, \qquad b = \|\underline{B}\|, \qquad \nu_1 = \|\underline{X}^{(1)}\|, \qquad \nu_2 = \|\underline{X}^{(2)}\|$$

Let

$$t = \nu_1^2 + \nu_2^2$$

1. n = m

$$g(t) = \frac{1}{2\sigma_1^2}\left(\frac{t}{a^2}\right)^{(n-1)/2}\left(\frac{\sigma_1}{\sigma_2}\right)^n \exp\left(-\frac{t}{2\sigma_1^2}\right)\exp\left(-\frac{1}{2}\left(\frac{a^2}{\sigma_1^2}+\frac{b^2}{\sigma_2^2}\right)\right)$$

$$\cdot \sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+\ell+j\right)}{k!\ j!\ \Gamma\left(\frac{n}{2}+j\right)}\left(\frac{\sqrt{t}\ b^2\sigma_1^2}{2a\sigma_2^4}\right)^j\left(\frac{\sqrt{t}\ \left(\sigma_2^2-\sigma_1^2\right)}{a\sigma_2^2}\right)^{\ell}$$

$$\cdot\ I_{n+j+\ell-1}\left(\frac{\sqrt{t}\ a}{\sigma_1^2}\right) \qquad t \geq 0$$

$$\Psi_t(\mu) = \left(\frac{1}{\left(1-2i\mu\sigma_1^2\right)\left(1-2i\mu\sigma_2^2\right)}\right)^{n/2}\exp\left(\frac{i\mu a^2}{1-2i\mu\sigma_1^2}\right)\exp\left(\frac{i\mu b^2}{1-2i\mu\sigma_2^2}\right)$$

2. <u>n even, m even</u>

$$g(t) = \frac{1}{2\sigma_1^2}\left(\frac{t}{a^2}\right)^{(n+m-2)/4}\left(\frac{\sigma_1}{\sigma_2}\right)^m \exp\left(-\frac{t}{2\sigma_1^2}\right)\exp\left(-\frac{1}{2}\left(\frac{a^2}{\sigma_1^2}+\frac{b^2}{\sigma_2^2}\right)\right)$$

$$\cdot \sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{\Gamma\left(\frac{m}{2}+j+\ell\right)}{j!\;\ell!\;\Gamma\left(\frac{m}{2}+j\right)}\left(\frac{\sqrt{t}\,b^2\sigma_1^2}{2a\sigma_2^2}\right)^j\left(\frac{\sqrt{t}\,\left(\sigma_2^2-\sigma_1^2\right)}{a\sigma_2^2}\right)^\ell$$

$$\cdot I_{[(n+m)/2]+j+\ell-1}\left(\frac{\sqrt{t}\,a}{\sigma_1^2}\right) \qquad t \geqq 0$$

[Ref. 24, p. 59]

$$\Psi_t(\mu) = \left(\frac{1}{1-2i\mu\sigma_1^2}\right)^{n/2}\left(\frac{1}{1-2i\mu\sigma_2^2}\right)^{m/2}\exp\left(\frac{i\mu a^2}{1-2i\mu\sigma_1^2}\right)\exp\left(\frac{i\mu b^2}{1-2i\mu\sigma_2^2}\right)$$

3. <u>n, m</u>

Same as 2.

IV. DIFFERENCE

A. CENTRAL CHI SQUARE (-) CENTRAL CHI SQUARE

1. Independent

$$\underline{X}^{(1)} \in N_n\left(\underline{0},\sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{0},\sigma_2^2\right)$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let $$s = r_1^2 - r_2^2$$

a. $n = m = 1$

$$g(s) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{\left(\sigma_2^2 - \sigma_1^2\right)s}{4\sigma_1^2\sigma_2^2}\right) K_0\left(\frac{\left(\sigma_1^2 + \sigma_2^2\right)s}{4\sigma_1^2\sigma_2^2}\right)$$

$$\psi_s(\mu) = \left(\frac{1}{\left(1 - 2i\mu\sigma_1^2\right)\left(1 + 2i\mu\sigma_2^2\right)}\right)^{1/2}$$

b. $n = m = 2$

$$g(s) = \frac{1}{2\left(\sigma_1^2 + \sigma_2^2\right)} \exp\left(-\frac{s\left(\sigma_2^2 - \sigma_1^2\right) + |s|\left(\sigma_1^2 + \sigma_2^2\right)}{4\sigma_1^2\sigma_2^2}\right)$$

$$\psi_s(\mu) = \frac{1}{\left(1 - 2i\mu\sigma_1^2\right)\left(1 + 2i\mu\sigma_2^2\right)}$$

c. $n = m = 2k$

$$g(s) = \frac{1}{2\left(\sigma_1^2 + \sigma_2^2\right)(k-1)!}\left(\frac{|s|}{2\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{k-1} \exp\left(-\frac{s\left(\sigma_2^2 - \sigma_1^2\right) + |s|\left(\sigma_1^2 + \sigma_2^2\right)}{4\sigma_1^2\sigma_2^2}\right)$$

$$\cdot \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j!\ (k-j-1)!}\left(\frac{2\sigma_1^2\sigma_2^2}{|s|\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{j}$$

$$\psi_s(\mu) = \left(\frac{1}{\left(1 - 2i\mu\sigma_1^2\right)\left(1 + 2i\mu\sigma_2^2\right)}\right)^{k}$$

d. $n = m$

$$g(s) = \frac{1}{2\sqrt{\pi\sigma_1^2\sigma_2^2}\ \Gamma\left(\frac{n}{2}\right)}\left(\frac{|s|}{2\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{(n-1)/2} \exp\left(-\frac{\left(\sigma_2^2 - \sigma_1^2\right)s}{4\sigma_1^2\sigma_2^2}\right)$$

$$\cdot K_{(n-1)/2}\left(\frac{\left(\sigma_1^2 + \sigma_2^2\right)|s|}{4\sigma_1^2\sigma_2^2}\right)$$

$$\psi_s(\mu) = \left(\frac{1}{\left(1 - 2i\mu\sigma_1^2\right)\left(1 + 2i\mu\sigma_2^2\right)}\right)^{n/2}$$

e. $n = 2k, \quad m = 2$

$$g(s) = \begin{cases} \dfrac{(-s)^{k-(1/2)} \exp\left(\dfrac{s}{2\sigma_2^2}\right)}{2^{k+(1/2)}\,\sigma_1^{2k-1}\,\sigma_2\left(\sigma_1^2 + \sigma_2^2\right)^{1/2}\,[(k-1)!]^2} \displaystyle\sum_{j=0}^{k} \frac{k!}{(k-j)!} \cdot \left(-\frac{2\sigma_1^2\sigma_2^2}{s\left(\sigma_1^2 + \sigma_2^2\right)}\right)^j & s \leq 0 \\[2em] \dfrac{\sigma_1\,\sigma_2^{2k\cdot 1}\,\exp\left(-\dfrac{s}{2\sigma_1^2}\right)}{(2s)^{1/2}\left(\sigma_1^2 + \sigma_2^2\right)^{k+(1/2)}} & s \geq 0 \end{cases}$$

$$\psi_s(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2}\right)\left(\frac{1}{1 + 2i\mu\sigma_2^2}\right)^k$$

f. n even, m even

$$
g(s) = \begin{cases}
\dfrac{(-s)^{(m+n-4)/4} \exp\left(-\dfrac{\left(\sigma_2^2 - \sigma_1^2\right)s}{4\sigma_1^2\sigma_2^2}\right)}{2^{(m+n)/4}\, \sigma_1^n\, \sigma_2^m \left(\dfrac{1}{\sigma_1^2} + \dfrac{1}{\sigma_2^2}\right)^{(n+m)/4} \Gamma\left(\dfrac{m}{2}\right)} \\
\qquad \cdot\, W_{(m-n)/4(n+m-2)/4}\left(-\dfrac{s\left(\sigma_1^2 + \sigma_2^2\right)}{2\sigma_1^2\sigma_2^2}\right) & s < 0 \\
\dfrac{s^{(m+n-4)/4} \exp\left(-\dfrac{\left(\sigma_2^2 - \sigma_1^2\right)s}{4\sigma_1^2\sigma_2^2}\right)}{2^{(m+n)/4}\, \sigma_1^n\, \sigma_2^m \left(\dfrac{1}{\sigma_1^2} + \dfrac{1}{\sigma_2^2}\right)^{(m+n)/4} \Gamma\left(\dfrac{n}{2}\right)} \\
\qquad \cdot\, W_{(n-m)/4,(n+m-2)/4}\left(\dfrac{s\left(\sigma_1^2 + \sigma_2^2\right)}{2\sigma_1^2\sigma_2^2}\right) & s \geqq 0
\end{cases}
$$

[Ref. 24, p. 65]

$$
s(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2}\right)^{n/2} \left(\frac{1}{1 + 2i\mu\sigma_2^2}\right)^{m/2}
$$

g. n, m

Same as f.

2. Dependent

$$\underline{X}^{(1)} \in N_n\left(\underline{0}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_n\left(\underline{0}, \sigma_2^2\right)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \qquad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let $s = r_1^2 - r_2^2, \qquad \gamma = \left[(w_{11} + w_{22})^2 - 4w_{12}^2\right]^{1/2}$

a. $n = 1$

$$g(s) = \frac{\exp\left(-\frac{(w_{11} - w_{22})s}{4}\right)}{2\pi|\underline{M}|^{1/2}} K_0\left(\frac{|s|\gamma}{4}\right)$$

$$\psi_s(\mu) = \left(\frac{|\underline{M}|}{\left(\sigma_2^2 - 2i\mu|\underline{M}|\right)\left(\sigma_1^2 + 2i\mu|\underline{M}|\right) - \rho^2\sigma_1^2\sigma_2^2}\right)^{1/2}$$

b. $n = 2$

$$g(s) = \begin{cases} \dfrac{2\exp\left(\frac{\alpha s}{4}\right)}{|\underline{M}|\ \gamma\alpha} & s < 0 \\ \\ 1 - \dfrac{2\exp\left(\frac{\beta s}{4}\right)}{|\underline{M}|\ \gamma\beta} & \end{cases}$$

$$\alpha = \gamma - (w_{11} - w_{22}), \qquad \beta = \gamma + (w_{11} - w_{22})$$

$$\psi_s(\mu) = \frac{|\underline{M}|}{\left(\sigma_2^2 - 2i\mu|\underline{M}|\right)\left(\sigma_1^2 + 2i\mu|\underline{M}|\right) - \rho^2\sigma_1^2\sigma_2^2}$$

c. $n = 2k$

$$g(s) = \frac{|s|^{k-1} \exp\left(-\frac{(w_{11} - w_{22})s}{4}\right)}{(k-1)!\ 2^k\ |\underline{M}|^k\ \gamma^k}$$

$$\cdot \exp\left(-\frac{\gamma|s|}{4}\right) \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j!\ (k-j-1)!} \left(\frac{2}{|s|\gamma}\right)^j$$

$$\psi_s(\mu) = \left(\frac{|\underline{M}|}{\left(\sigma_2^2 - 2i\mu|\underline{M}|\right)\left(\sigma_1^2 + 2i\mu|\underline{M}|\right) - \rho^2\sigma_1^2\sigma_2^2}\right)^k$$

d. n

$$g(s) = \frac{|s|^{(n-1)/2} \exp\left(-\frac{(w_{11} - w_{22})s}{4}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right) 2^{(n+1)/2}\ |\underline{M}|^{n/2}\ \gamma^{(n-1)/2}}\ K_{(n-1)/2}\left(\frac{\gamma|s|}{4}\right)$$

[Ref. 24, p. 61]

$$\psi_s(\mu) = \left(\frac{|\underline{M}|}{\left(\sigma_2^2 - 2i\mu|\underline{M}|\right)\left(\sigma_1^2 + 2i\mu|\underline{M}|\right) - \rho^2\sigma_1^2\sigma_2^2}\right)^{n/2}$$

B. NON CENTRAL CHI SQUARE (-) CENTRAL CHI SQUARE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n\left(\underline{A},\sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{0},\sigma_2^2\right)$$

$$a = \|\underline{A}\|, \qquad \nu = \|\underline{X}^{(1)}\|, \qquad r = \|\underline{X}^{(2)}\|$$

Let $$w = \nu^2 - r^2$$

1. $\underline{n = 1, \quad m = 1}$

$$g(w) = \begin{cases} \dfrac{\exp\left(-\dfrac{a^2}{2\sigma_1^2}\right)\exp\left(-\dfrac{\left(\sigma_2^2 - \sigma_1^2\right)w}{4\sigma_1^2\sigma_2^2}\right)}{\pi\left[-2w\left(\sigma_1^2 + \sigma_2^2\right)\right]^{1/2}} \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{-wa^4\sigma_2^2}{8\sigma_1^6\left(\sigma_2^2 + \sigma_1^2\right)}\right)^{j/2} \\ \qquad \cdot W_{-k/2,k/2}\left(-\dfrac{\left(\sigma_1^2 + \sigma_2^2\right)w}{2\sigma_1^2\sigma_2^2}\right) & w < 0 \\ \dfrac{\exp\left(-\dfrac{a^2}{2\sigma_1^2}\right)\exp\left(-\dfrac{\left(\sigma_2^2 - \sigma_1^2\right)w}{4\sigma_1^2\sigma_2^2}\right)}{\left[2w\left(\sigma_1^2 + \sigma_2^2\right)\right]^{1/2}} \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!\,\Gamma\left(j + \frac{1}{2}\right)}\left(\frac{wa^4\sigma_2^2}{8\sigma_1^6\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{j/2} \\ \qquad \cdot W_{k/2,k/2}\left(\dfrac{\left(\sigma_1^2 + \sigma_2^2\right)w}{2\sigma_1^2\sigma_2^2}\right) & w \geq 0 \end{cases}$$

$$\psi_w(\mu) = \left(\frac{1}{\left(1 - 2i\mu\sigma_1^2\right)\left(1 + 2i\mu\sigma_2^2\right)}\right)^{1/2} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right)$$

2. <u>$n = 2, \quad m = 2$</u>

$$g(w) = \begin{cases} \dfrac{c}{2\sigma_2^2} \exp\left(\dfrac{w}{2\sigma_2^2}\right) & w \leq 0 \\[2ex] \dfrac{c}{2\sigma_2^2} \exp\left(\dfrac{w}{2\sigma_2^2}\right) Q\left(\left(\dfrac{a^2\sigma_2^2}{\sigma_1^2\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{1/2} ; \left(\dfrac{w\left(\sigma_1^2 + \sigma_2^2\right)}{\sigma_1^2\sigma_2^2}\right)^{1/2}\right) & w > 0 \end{cases}$$

[Ref. 23]

where

$$c = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \exp\left(- \frac{a^2\left(\sigma_2^2 + 2\sigma_1^2\right)}{4\sigma_1^2\left(\sigma_1^2 + \sigma_2^2\right)}\right)$$

$$\psi_w(\mu) = \frac{1}{\left(1 - 2i\mu\sigma_1^2\right)\left(1 + 2i\mu\sigma_2^2\right)} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right)$$

3. n = m

$$
g(w) = \begin{cases}
\dfrac{(-w)^{(n-2)/2} \exp\left(-\dfrac{a^2}{2\sigma_1^2}\right) \exp\left(-\dfrac{\left(\sigma_2^2 - \sigma_1^2\right) w}{4\sigma_1^2\sigma_2^2}\right)}{2^{n/2}\left(\sigma_1^2 + \sigma_2^2\right)^{n/2} \Gamma\left(\frac{n}{2}\right)} \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!} \\
\quad \cdot \left(\dfrac{-wa^4\sigma_2^2}{8\sigma_1^6\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{j/2} W_{-j/2,(n+j-2)/2}\left(-\dfrac{\left(\sigma_1^2 + \sigma_2^2\right) w}{2\sigma_1^2\sigma_2^2}\right) & w < 0 \\
\dfrac{w^{(n-2)/2} \exp\left(-\dfrac{a^2}{2\sigma_1^2}\right) \exp\left(-\dfrac{\left(\sigma_2^2 - \sigma_1^2\right) w}{4\sigma_1^2\sigma_2^2}\right)}{2^{n/2}\left(\sigma_1^2 + \sigma_2^2\right)^{n/2}} \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!\,\Gamma\left(\frac{n}{2} + j\right)} \\
\quad \cdot \left(\dfrac{wa^4\sigma_2^2}{8\sigma_1^6\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{j/2} W_{j/2,(n+j-2)/2}\left(\dfrac{\left(\sigma_1^2 + \sigma_2^2\right) w}{2\sigma_1^2\sigma_2^2}\right) & w \geqq 0
\end{cases}
$$

$$
\psi_w(\mu) = \left(\frac{1}{\left(1 - 2i\mu\sigma_1^2\right)\left(1 + 2i\mu\sigma_2^2\right)}\right)^{n/2} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right)
$$

4. $\underline{n = 2, \quad m = 2k}$

$$
g(w) = \begin{cases}
\dfrac{(-w)^{(k-1)/2} \exp\left(-\dfrac{a^2}{2\sigma_1^2}\right) \exp\left(-\dfrac{\left(\sigma_2^2 - \sigma_1^2\right)w}{4\sigma_1^2\sigma_2^2}\right)}{2^{(k+1)/2}\sigma_1^2\sigma_2^{2k}\left(\dfrac{1}{\sigma_1^2} + \dfrac{1}{\sigma_2^2}\right)^{(k+1)/2}} \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!(k-1)!} \\
\quad \cdot \left(\dfrac{-wa^4\sigma_2^2}{8\sigma_1^6\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{j/2} W_{(k-j-1)/2,(k+j)/2}\left(-\dfrac{\left(\sigma_1^2 + \sigma_2^2\right)w}{2\sigma_1^2\sigma_2^2}\right) & w < 0 \\
\dfrac{w^{(k-1)/2} \exp\left(-\dfrac{a^2}{2\sigma_1^2}\right) \exp\left(-\dfrac{\left(\sigma_2^2 - \sigma_1^2\right)w}{4\sigma_1^2\sigma_2^2}\right)}{2^{(k+1)/2}\sigma_1^2\sigma_2^{2k}\left(\dfrac{1}{\sigma_1^2} + \dfrac{1}{\sigma_2^2}\right)^{(k+1)/2}} \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!\ j!} \\
\quad \cdot \left(\dfrac{wa^4\sigma_2^2}{8\sigma_1^6\left(\sigma_i^2 + \sigma_2^2\right)}\right)^{j/2} W_{(1+j-k)/2,(k+j)/2}\left(\dfrac{\left(\sigma_1^2 + \sigma_2^2\right)w}{2\sigma_1^2\sigma_2^2}\right) & w \geqq 0
\end{cases}
$$

$$
\psi_w(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2}\right)\left(\frac{1}{1 + 2i\mu\sigma_2^2}\right)^k \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right)
$$

5. $n = 2k, \quad m = 2$

$$
g(w) = \begin{cases} \dfrac{d}{2\sigma_2^2} \exp\left(\dfrac{w}{2\sigma_2}\right) & w \leqq 0 \\[2ex] \dfrac{d}{2\sigma_2^2} \exp\left(\dfrac{w}{2\sigma_2}\right) Q_k\left(\left(\dfrac{a^2\sigma_2^2}{2\sigma_1^2\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{1/2} ; \left(\dfrac{w\left(\sigma_1^2 + \sigma_2^2\right)}{\sigma_1^2\sigma_2^2}\right)^{1/2}\right) & w \geqq 0 \end{cases}
$$

$$
\psi(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2}\right)^k \left(\frac{1}{1 + 2i\mu\sigma_2^2}\right) \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right)
$$

where

$$
d = \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)^k \exp\left(- \frac{a^2\left(\sigma_2^2 + 2\sigma_1^2\right)}{4\sigma_1^2\left(\sigma_1^2 + \sigma_2^2\right)}\right)
$$

6. <u>n even, m even</u>

$$g(w) = \begin{cases} \dfrac{(-w)^{(n+m-4)/4} \exp\left(- \dfrac{a^2}{2\sigma_1^2}\right) \exp\left(- \dfrac{\left(\sigma_2^2 - \sigma_1^2\right)w}{4\sigma_1^2\sigma_2^2}\right)}{2^{(m+n)/4}\, \sigma_1^n \sigma_2^m \left(\dfrac{1}{\sigma_1^2} + \dfrac{1}{\sigma_2^2}\right)^{(m+n)/4}} \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!\, \Gamma\left(\frac{m}{2}\right)} \\ \cdot \left(\dfrac{-wa^4\sigma_2^2}{8\sigma_1^6\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{j/2} W_{(m-n-2k)/4,(n+m+2k-2)/4} \left(- \dfrac{\left(\sigma_1^2 + \sigma_2^2\right)w}{2\sigma_1^2\sigma_2^2}\right) & w < 0 \\ \dfrac{w^{(n+m-4)/2} \exp\left(- \dfrac{a^2}{2\sigma_1^2}\right) \exp\left(- \dfrac{\left(\sigma_2^2 - \sigma_1^2\right)w}{4\sigma_1^2\sigma_2^2}\right)}{2^{(m+n)/4}\, \sigma_1^n \sigma_2^m \left(\dfrac{1}{\sigma_1^2} + \dfrac{1}{\sigma_2^2}\right)^{(m+n)/4}} \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!\, \Gamma\left(\frac{n}{2} + j\right)} \\ \cdot \left(\dfrac{wa^4\sigma_2^2}{8\sigma_1^6\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{j/2} W_{(n-m+2k)/4,(n+m+2k-2)/4} \left(\dfrac{\left(\sigma_1^2 + \sigma_2^2\right)w}{2\sigma_1^2\sigma_2^2}\right) & w \geq 0 \end{cases}$$

[Ref. 24, p. 64]

$$\psi_w(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2}\right)^{n/2} \left(\frac{1}{1 + 2i\mu\sigma_2^2}\right)^{m/2} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right)$$

7. <u>n, m</u>

Same as 6.

C. NON CENTRAL CHI SQUARE (-) NON CENTRAL CHI SQUARE

$$\underline{X}^{(1)} \in N_n\left(\underline{A}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{B}, \sigma_2^2\right)$$

$$a = \|\underline{A}\|, \qquad b = \|\underline{B}\|, \qquad \nu_1 = \|\underline{X}^{(1)}\|, \qquad \nu_2 = \|\underline{X}^{(2)}\|$$

Let

$$t = \nu_1^2 - \nu_2^2$$

$$g(t) = \begin{cases} \dfrac{(-t)^{(m+n-4)/4} \exp\left[-\frac{1}{2}\left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2}\right)\right] \exp\left(-\frac{\left(\sigma_2^2 - \sigma_1^2\right)t}{4\sigma_1^2\sigma_2^2}\right)}{2^{(m+n)/4}\, \sigma_1^n \sigma_2^m \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{(n+m)/4}} \displaystyle\sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \\[2ex] \quad \cdot \dfrac{1}{j!\ \ell!\ \Gamma\left(\frac{m}{2} + \ell\right)} \left(\dfrac{-ta^4\sigma_2^2}{8\sigma_1^6\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{j/2} \left(\dfrac{-tb^4\sigma_1^2}{8\sigma_2^6\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{\ell/2} \\[2ex] \quad \cdot W_{[(m-n)/4]+[(\ell-j)/2];[(n+m-2)/4]+[(j+\ell)/2]}\left(-\dfrac{\left(\sigma_2^2 - \sigma_1^2\right)t}{2\sigma_1^2\sigma_2^2}\right) & t < 0 \\[3ex] \dfrac{t^{(m+n-4)/4} \exp\left[-\frac{1}{2}\left(\frac{a^2}{\sigma_1^2} + \frac{b^2}{\sigma_2^2}\right)\right] \exp\left(-\frac{\left(\sigma_2^2 - \sigma_1^2\right)t}{4\sigma_1^2\sigma_2^2}\right)}{2^{(m+n)/4}\, \sigma_1^n \sigma_2^m \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{(n+m)/4}} \displaystyle\sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \\[2ex] \quad \cdot \dfrac{1}{j!\ \ell!\ \Gamma\left(\frac{n}{2} + j\right)} \left(\dfrac{ta^4\sigma_2^2}{8\sigma_1^6\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{j/2} \left(\dfrac{tb^4\sigma_1^2}{8\sigma_2^6\left(\sigma_1^2 + \sigma_2^2\right)}\right)^{\ell/2} \\[2ex] \quad \cdot W_{[(n-m)/4]+[(j-\ell)/2],[(m+n-2)/4]+[(j+\ell)/2]}\left(\dfrac{\left(\sigma_2^2 - \sigma_1^2\right)t}{2\sigma_1^2\sigma_2^2}\right) & t \geq 0 \end{cases}$$

[Ref. 24, p. 63]

$$\psi_t(\mu) = \left(\frac{1}{1 - 2i\mu\sigma_1^2}\right)^{n/2} \left(\frac{1}{1 + 2i\mu\sigma_2^2}\right)^{m/2} \exp\left(\frac{i\mu a^2}{1 - 2i\mu\sigma_1^2}\right) \exp\left(- \frac{i\mu b^2}{1 + 2i\mu\sigma_2^2}\right)$$

V. PRODUCT

A. GAUSSIAN WITH ZERO MEANS

1. Independent

$$\underline{X}^{(1)} \in N_n\left(\underline{0}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_n\left(\underline{0}, \sigma_2^2\right)$$

Let $z = \left(\underline{X}^{(1)}, \underline{X}^{(2)}\right) = \sum_{k=1}^{n} x_k^{(1)} x_k^{(2)}; \qquad c = \sigma_1 \sigma_2$

a. $n = 1$

$$f(z) = \frac{1}{\pi c} K_0\left(\frac{|z|}{c}\right) \qquad \text{[Ref. 24, p. 45]}$$

$$\psi_z(\mu) = \left(\frac{1}{1 + c^2 \mu^2}\right)^{1/2}$$

b. $n = 2$

$$f(z) = \frac{1}{2c} \exp\left(- \frac{|z|}{c}\right)$$

$$\psi_z(\mu) = \frac{1}{1 + c^2 \mu^2}$$

c. $n = 2k$

$$f(z) = \frac{|z|^{k-1} \exp\left(- \frac{|z|}{c}\right)}{2^{k-2} c^k (k-1)!} \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j!\ (k-j-1)!} \left(\frac{c}{2|z|}\right)^j$$

$$\psi_z(\mu) = \left(\frac{1}{1 + c^2\mu^2}\right)^k$$

d. n

$$f(z) = \frac{|z|^{(n-1)/2}}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right) 2^{(n-1)/2} c^{(n+1)/2}} K_{(n-1)/2}\left(\frac{|z|}{c}\right)$$

[Ref. 24, p. 42]

$$\psi_z(\mu) = \left(\frac{1}{1 + c^2\mu^2}\right)^{n/2}$$

e. Angle $z = \left(\underline{X}^{(1)}, \underline{X}^{(2)}\right) = \|\underline{X}^{(1)}\| \, \|\underline{X}^{(2)}\| \cos \phi$

Phi (ϕ) is the angle between $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$.

$$\pi(\phi) = \frac{(\sin \phi)^{n-2}}{B\left(\frac{1}{2}, \frac{n-1}{2}\right)} \qquad 0 < \phi < 2\pi$$

[Ref. 24, p. 47]

2. Dependent

$$\underline{X}^{(1)} \in N_n\left(\underline{0}, \sigma_1^2\right), \qquad \underline{X}^{(2)} = N_n\left(\underline{0}, \sigma_2^2\right)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \qquad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$z = \left(\underline{X}^{(1)}, \underline{X}^{(2)}\right) = \sum_{k=1}^{n} x_k^{(1)} x_k^{(2)}$$

a. $n = 1$

$$f(z) = \frac{1}{\pi} |\underline{W}|^{1/2} \exp(-w_{12}z)\, K_0\left(|z|\sqrt{w_{11}w_{22}}\right)$$

[Ref. 24, p. 45]

$$\psi_z(\mu) = \left(\frac{1}{1 - 2i\mu\rho c + \mu^2 c^2 (1-\rho^2)}\right)^{1/2}$$

b. $n = 2$

$$f(z) = \frac{\exp(-w_{12}z)}{2c} \exp\left(-|z|\sqrt{w_{11}w_{22}}\right)$$

$$\psi_z(\mu) = \frac{1}{1 - 2i\mu\rho c + \mu^2 c^2 (1-\rho^2)}$$

c. $n = 2k$

$$f(z) = \frac{|z|^{k-1} \exp(-w_{12}z) \exp\left(-|z|\sqrt{w_{11}w_{22}}\right)}{(k-1)!\, 2^k\, c^k} \sum_{j=0}^{k-1} \frac{(k+j-1)!}{j!\,(k-j-1)!}$$

$$\cdot \left(\frac{1}{2|z|\sqrt{w_{11}w_{22}}}\right)^j$$

$$\psi_z(\mu) = \left(\frac{1}{1 - 2i\mu\rho c + \mu^2 c^2 (1-\rho^2)}\right)^k$$

d. n

$$f(z) = \frac{|z|^{(n-1)/2} \exp(-w_{12}z)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right) 2^{(n-1)/2} |\underline{M}|^{1/2} c^{(n-1)/2}} K_{(n-1)/2}\left(|z|\sqrt{w_{11}w_{22}}\right)$$

[Ref. 24, p. 42]

$$\psi_z(\mu) = \left(\frac{1}{1 - 2i\mu\rho c + \mu^2 c^2 (1-\rho^2)}\right)^{n/2}$$

e. Angle $z = \left(\underline{X}^{(1)}, \underline{X}^{(2)}\right) = \|\underline{X}^{(1)}\| \, \|\underline{X}^{(2)}\| \cos\phi$

Phi (ϕ) is the angle between $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$.

$$\pi(\phi) = \frac{(n-1)\,\Gamma(n)\,(\sin\phi)^{n-2}}{|\underline{M}|^{n/2}\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)\left(\sqrt{w_{11}w_{22}} + w_{12}\cos\phi\right)^n}$$

$$\cdot\ {}_2F_1\left(n, \frac{1}{2}, n+\frac{1}{2}; \frac{w_{12}\cos\phi - \sqrt{w_{11}w_{22}}}{w_{12}\cos\phi + \sqrt{w_{11}w_{22}}}\right)$$

[Ref. 24, p. 45]

B. GAUSSIAN WITH ONE NON ZERO MEAN (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n(\underline{A}, \sigma^2), \qquad \underline{X}^{(2)} \in N_n(\underline{0}, \sigma^2), \qquad a = \|\underline{A}\|$$

Let $$z = \left(\underline{X}^{(1)}, \underline{X}^{(2)}\right) = \sum_{k=1}^{n} x_k^{(1)} x_k^{(2)}$$

1. $\underline{n = 1}$

$$f(z) = \frac{1}{\sqrt{\pi}\,\sigma^2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \sum_{j=0}^{\infty} \frac{1}{j!\,\Gamma\left(j+\frac{1}{2}\right)} \left(\frac{a^2|z|}{4\sigma^4}\right)^j K_j\left(\frac{|z|}{\sigma^2}\right)$$

$$\psi_z(\mu) = \left(\frac{1}{1+\sigma^4\mu^2}\right)^{1/2} \exp\left(-\frac{\mu^2\sigma^2 a^2}{2(1+\sigma^4\mu^2)}\right)$$

2. $\underline{n = 2}$

$$f(z) = \frac{1}{2\sigma^2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \exp\left(-\frac{|z|}{\sigma^2}\right) \sum_{j=0}^{\infty} \sum_{\ell=0}^{j}$$

$$\cdot \frac{(j+\ell)!}{2^{\ell}\, j!\, j!\, \ell!\, (j-\ell)!} \left(\frac{a^2}{4\sigma^2}\right)^j \left(\frac{|z|}{\sigma^2}\right)^{j-\ell}$$

$$\psi_z(\mu) = \frac{1}{1+\sigma^4\mu^2} \exp\left(-\frac{\mu^2\sigma^2 a^2}{2(1+\sigma^4\mu^2)}\right)$$

3. $\underline{n = 2k \ \text{(even)}}$

$$f(z) = \frac{1}{2\sigma^2} \left(\frac{|z|}{2\sigma^2}\right)^{k-1} \exp\left(-\frac{a^2}{2\sigma^2}\right) \exp\left(-\frac{|z|}{\sigma^2}\right) \sum_{j=0}^{\infty} \sum_{\ell=0}^{k+j-1}$$

$$\cdot \frac{(k+j+\ell-1)!}{2^{\ell}\, j!\, (k+j-1)!\, \ell!\, (k+j-\ell-1)!} \left(\frac{a^2}{4\sigma^2}\right)^j \left(\frac{|z|}{\sigma^2}\right)^{j-\ell}$$

$$\psi_z(\mu) = \left(\frac{1}{1 + \sigma^4\mu^2}\right)^k \exp\left(- \frac{\mu^2\sigma^2 a^2}{2(1 + \sigma^4\mu^2)}\right)$$

4. $\underline{n}$

$$f(z) = \frac{1}{\sqrt{\pi}\,\sigma^2}\left(\frac{z}{2\sigma^2}\right)^{(n-1)/2} \exp\left(- \frac{a^2}{2\sigma^2}\right)\sum_{j=0}^{\infty} \frac{1}{j!\,\Gamma\left(\frac{n}{2} + j\right)}$$

$$\cdot\left(\frac{a^2|z|}{2\sigma^2}\right)^{[(n-1)/2]+j} K_{[(n-1)/2]+j}\left(\frac{|z|}{\sigma^2}\right)$$

[Ref. 32]

$$\psi_z(\mu) = \left(\frac{1}{1 + \mu^2\sigma^4}\right)^{n/2} \exp\left(- \frac{\mu^2\sigma^2 a^2}{2(1 + \mu^2\sigma^4)}\right)$$

C. GAUSSIAN WITH NON ZERO MEANS

1. Independent

$$\underline{X}^{(1)} \in N_n(\underline{A},\sigma^2), \qquad \underline{X}^{(2)} \in \underline{N}_n(\underline{B},\sigma^2)$$

$$a = \|\underline{A}\|, \qquad b = \|\underline{B}\|$$

Let $$z = \left(\underline{X}^{(1)}, \underline{X}^{(2)}\right) = \sum_{k=0}^{n} x_k^{(1)} x_k^{(2)}$$

$$\alpha^2 = \frac{1}{4}\left((\underline{A}+\underline{B}),\ (\underline{A}+\underline{B})\right) = \frac{a^2+b^2}{4} + \frac{1}{2}\sum_{k=1}^{n} a_k b_k$$

$$\beta^2 = \frac{1}{4}\left((\underline{A}-\underline{B}),\ (\underline{A}-\underline{B})\right) = \frac{a^2+b^2}{4} - \frac{1}{2}\sum_{k=1}^{n} a_k b_k$$

$$f(z) = \begin{cases} \dfrac{(-z)^{(n-2)/2}\exp\left(-\dfrac{a^2+b^2}{2\sigma^2}\right)}{2^{n/2}\sigma^n} \displaystyle\sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{1}{j!\ \ell!\ \Gamma\left(\frac{n}{2}+\ell\right)} \\ \quad \cdot \left(\dfrac{-\alpha^4 z}{2\sigma^6}\right)^{j/2}\left(\dfrac{-\beta^4 z}{2\sigma^6}\right)^{\ell/2} W_{(\ell-j)/2,(n+\ell+j-1)/2}\left(\dfrac{-2z}{\sigma^2}\right) & z < 0 \\ \dfrac{z^{(n-2)/2}\exp\left(-\dfrac{a^2+b^2}{2\sigma^2}\right)}{2^{n/2}\sigma^n} \displaystyle\sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{1}{j!\ \ell!\ \Gamma\left(\frac{n}{2}+j\right)} \\ \quad \cdot \left(\dfrac{\alpha^4 z}{2\sigma^6}\right)^{j/2}\left(\dfrac{\beta^4 z}{2\sigma^6}\right)^{\ell/2} W_{(j-\ell)/2,(n+j+\ell-1)/2}\left(\dfrac{2z}{\sigma^2}\right) & z \geq 0 \end{cases}$$

[Ref. 24, p. 63]

$$\psi_z(\mu) = \left(\frac{1}{1+\sigma^2\mu^2}\right)^{n/2} \exp\left(-\frac{(a^2+b^2)\sigma^2\mu^2 - 2i\mu\displaystyle\sum_{r=0}^{n} a_r b_r}{1+\sigma^4\mu^2}\right)$$

2. Dependent

$$\underline{X}^{(1)} \in N_n(\underline{A},\sigma^2), \qquad \underline{X}^{(2)} \in N_n(\underline{B},\sigma^2)$$

$$a = \|\underline{A}\|, \qquad b = \|\underline{B}\|, \qquad M = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}; \qquad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

Let $$z = \left(\underline{X}^{(1)}, \underline{X}^{(2)}\right) = \sum_{k=1}^{n} x_k^{(1)} x_k^{(2)}$$

$$\alpha^2 = \frac{1}{4}\left((\underline{A}+\underline{B}), (\underline{A}+\underline{B})\right) = \frac{a^2+b^2}{4} + \frac{1}{2}\sum_{k=1}^{n} a_k b_k$$

$$\beta^2 = \frac{1}{4}\left((\underline{A}-\underline{B}), (\underline{A}-\underline{B})\right) = \frac{a^2+b^2}{4} - \frac{1}{2}\sum_{k=1}^{n} a_k b_k$$

$$f(z) = \begin{cases} \dfrac{(-z)^{(n-2)/2} \exp\left(-2\beta^2\sqrt{w_{11}w_{22}}\right)}{2^{n/2}\,\sigma^n} \exp(-w_{12}z) \displaystyle\sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{1}{j!\,\ell!\,\Gamma\left(\frac{n}{2}+\ell\right)} \\ \quad \cdot \left(-\dfrac{\alpha^4(1-\rho)z}{2(1+\rho)^3\sigma^6}\right)^{j/2} \left(-\dfrac{\beta^4(1+\rho)z}{2(1-\rho)^3\sigma^6}\right)^{\ell/2} \\ \quad \cdot W_{(\ell-j)/2,(n+j+\ell-1)/2}\left(-2z\sqrt{w_{11}w_{22}}\right) & z < 0 \\ \dfrac{z^{(n-2)/2} \exp\left(-2\beta^2\sqrt{w_{11}w_{22}}\right)}{2^{n/2}\,\sigma^n} \exp(-w_{12}z) \displaystyle\sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{1}{j!\,\ell!\,\Gamma\left(\frac{n}{2}+j\right)} \\ \quad \cdot \left(\dfrac{\alpha^4(1-\rho)z}{2(1+\rho)^3\sigma^6}\right)^{j/2} \left(\dfrac{\beta^4(1+\rho)z}{2(1-\rho^3\sigma^6}\right)^{\ell/2} \\ \quad \cdot W_{(j-\ell)/2,(n+j+\ell-1)/2}\left(2z\sqrt{w_{11}w_{22}}\right) & z \geq 0 \end{cases}$$

[Ref. 24, p. 63]

$$\psi_z(\mu) = \left(\frac{1}{\gamma}\right)^{n/2} \exp\left(-\frac{(a^2 + b^2)\sigma^2\mu^2 - 2(i\mu + \rho\sigma^2\mu^2)\sum_{r=0}^{n} a_r b_r}{\gamma}\right)$$

$$\gamma = 1 - 2i\mu\sigma^2\rho + \mu^2\sigma^4(1-\rho^2)$$

D. RAYLEIGH (×) RAYLEIGH

1. Independent

$$\underline{X}^{(1)} \in N_n\left(\underline{0},\sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{0},\sigma_2^2\right)$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let $\quad z = r_1 r_2, \qquad c = \sigma_1\sigma_2$

a. $n = m = 1$

$$f(z) = \frac{2}{\pi c} K_0\left(\frac{z}{c}\right) \qquad z \geqq 0$$

$$Ez^k = (2c)^k \frac{\Gamma^2\left(\frac{k+1}{2}\right)}{\pi} \qquad k \geqq 0$$

b. $n = m = 2$

$$f(z) = \frac{z}{c^2} K_0\left(\frac{z}{c}\right) \qquad z \geqq 0$$

$$Ez^k = (2c)^k \Gamma^2\left(\frac{k}{2} + 1\right) \qquad k \geqq 0$$

c. $n = m = 2k$

$$f(z) = \frac{4}{z}\left(\frac{z}{2c}\right)^{2k} \frac{1}{[(k-1)!]^2} K_0\left(\frac{z}{c}\right) \qquad z \geq 0$$

$$Ez^{\ell} = (2c)^{\ell} \frac{\Gamma^2\left(k + \frac{\ell}{2}\right)}{[(k-1)!]^2} \qquad \ell \geq 0$$

[Ref. 24, p. 73]

d. n, m

$$f(z) = \frac{4}{z}\left(\frac{z}{2c}\right)^{(m+n)/2} \frac{1}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} K_{(n-m)/2}\left(\frac{z}{c}\right) \qquad z \geq 0$$

[Ref. 24, p. 49]

2. Dependent

$$\underline{x}^{(1)} \in N_n\left(\underline{0}, \sigma_1^2\right), \qquad \underline{x}^{(2)} \in N_n\left(\underline{0}, \sigma_2^2\right)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}; \qquad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

$$r_1 = \|\underline{x}^{(1)}\|, \qquad r_2 = \|\underline{x}^{(2)}\|$$

Let $$z = r_1 r_2$$

$$f(z) = \frac{|\underline{W}|^{n/2}\, z^{n/2}}{(2|w_{12}|)^{(n-2)/2}\, \Gamma\left(\frac{n}{2}\right)}\, I_{(n/2)-1}\,(z|w_{12}|)\; K_0\,\left(z\sqrt{w_{11}w_{22}}\right) \qquad z \geq 0$$

[Ref. 24, p. 47]

$$Ez^k = \frac{2^k\, |\underline{W}|^{n/2}\, \Gamma^2\left(\frac{n+k}{2}\right)}{(w_{11}w_{22})^{(n+k)/2}\, \Gamma^2\left(\frac{n}{2}\right)}\; {}_2F_1\left(\frac{n+k}{2}, \frac{n+k}{2}, \frac{n}{2}; \frac{w_{12}^2}{w_{11}w_{22}}\right)$$

[Ref. 24, p. 73]

E. RAYLEIGH (×) RICE (INDEPENDENT)

$$\underline{X}^{(1)} \in N_n\left(\underline{A}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{0}, \sigma_2^2\right)$$

$$a = \|\underline{A}\|, \qquad \nu = \|\underline{X}^{(1)}\|, \qquad r = \|\underline{X}^{(2)}\|$$

Let $\qquad u = \nu r$

1. $\underline{n = 1, \quad m = 1}$

$$f(u) = \frac{2}{\sqrt{\pi}\,\sigma_1\sigma_2} \exp\left(-\frac{a^2}{2\sigma_1^2}\right) \sum_{j=0}^{\infty} \frac{1}{j!\,\Gamma\left(j + \frac{1}{2}\right)} \left(\frac{a}{2\sigma_1^2}\right)^{2j} \left(\frac{u\sigma_1}{\sigma_2}\right)^{j} K_j\left(\frac{u}{\sigma_1\sigma_2}\right)$$

$$u \geq 0$$

2. $\underline{n = 2, \quad m = 2}$

$$f(u) = \frac{u}{\sigma_1^2\sigma_2^2} \exp\left(-\frac{a^2}{2\sigma_1^2}\right) \sum_{j=0}^{\infty} \frac{1}{[j!]^2} \left(\frac{a}{2\sigma_1^2}\right)^{2j} \left(\frac{u\sigma_1}{\sigma_2}\right)^{j} K_j\left(\frac{u}{\sigma_1\sigma_2}\right) \qquad u \geq 0$$

3. n, m

$$f(u) = \frac{4}{u}\left(\frac{u}{2\sigma_1\sigma_2}\right)^{(m+n)/2} \frac{\exp\left(-\frac{a^2}{2\sigma_1^2}\right)}{\Gamma\left(\frac{m}{2}\right)} \sum_{j=0}^{\infty} \frac{1}{j!\,\Gamma\left(\frac{n}{2}+j\right)}\left(\frac{a}{2\sigma_1^2}\right)^{2j}\left(\frac{u\sigma_1}{\sigma_2}\right)^{j}$$

$$\cdot K_{[(n-m)/2]+j}\left(\frac{u}{\sigma_1\sigma_2}\right) \qquad u \geq 0$$

[Ref. 24, p. 48]

F. RICE (×) RICE

$$\underline{X}^{(1)} \in N_n\left(\underline{A}, \sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_m\left(\underline{B}, \sigma_2^2\right)$$

$$a = \|\underline{A}\|, \qquad b = \|\underline{B}\|, \qquad \nu_1 = \|\underline{X}^{(1)}\|, \qquad \nu_2 = \|\underline{X}^{(2)}\|$$

Let $$q = \nu_1\nu_2$$

$$f(q) = \frac{4}{q}\left(\frac{q}{2\sigma_1\sigma_2}\right)^{(n+m)/2} \exp\left(-\frac{1}{2}\left(\frac{a^2}{\sigma_1^2}+\frac{b^2}{\sigma_2^2}\right)\right) \sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{1}{j!\,\ell!\,\Gamma\left(\frac{n}{2}+j\right)\Gamma\left(\frac{m}{2}+\ell\right)}$$

$$\cdot \left(\frac{a\sqrt{q}}{2\sigma_1^2}\right)^{2j}\left(\frac{b\sqrt{q}}{2\sigma_2^2}\right)^{2\ell}\left(\frac{\sigma_1}{\sigma_2}\right)^{j-\ell} K_{[(n-m)/2]+j-\ell}\left(\frac{q}{\sigma_1\sigma_2}\right) \qquad q \geq 0$$

[Ref. 24, p. 48]

$$Eq^p = (2\sigma_1\sigma_2)^p \; \frac{\Gamma\left(\frac{n+p}{2}\right)\Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \; \exp\left(-\frac{1}{2}\left(\frac{a^2}{\sigma_1^2}+\frac{b^2}{\sigma_2^2}\right)\right)$$

$$\cdot \; {}_1F_1\left(\frac{n+p}{2};\frac{n}{2};\frac{a^2}{2\sigma_1^2}\right) {}_1F_1\left(\frac{m+p}{2};\frac{m}{2};\frac{b^2}{2\sigma_2^2}\right) \qquad p > 0$$

[Ref. 24]

G. JOINT DENSITY

$$\underline{X}^{(1)} \in N_n\left(0,\sigma_1^2\right), \qquad \underline{X}^{(2)} \in N_n\left(0,\sigma_2^2\right)$$

$$\underline{M} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \qquad \underline{W} = \underline{M}^{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}$$

Also

$$\underline{X}^{(3)} \in N_n\left(\underline{0},\sigma_1^2\right), \qquad \underline{X}^{(4)} \in N_n\left(\underline{0},\sigma_2^2\right), \qquad \underline{M}$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|, \qquad r_3 = \|\underline{X}^{(3)}\|, \qquad r_4 = \|\underline{X}^{(4)}\|$$

Let $\qquad z_1 = r_1 r_3, \qquad z_2 = r_2 r_4$

Here r_1 and r_3 are independent

r_2 and r_4 are independent

r_1 and r_2 are related by $\underline{M}$

r_3 and r_4 are related by $\underline{M}$

$$f(z_1,z_2) = \frac{|\underline{W}|^n \, (z_1 z_2)^{n-1}}{2^{2n-4}\, \Gamma^2\left(\frac{n}{2}\right)} \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{j!\ \ell!\ \Gamma\left(\frac{n}{2}+j\right)\Gamma\left(\frac{n}{2}+\ell\right)}$$

$$\cdot \left(\frac{z_1 z_2 w_{12}^2}{4}\right)^{j+\ell} K_{j-\ell}\,(z_1 w_{11})\, K_{j-\ell}\,(z_2 w_{22}) \qquad z_1, z_2 > 0$$

[Ref. 24, p. 49]

VI. GENERAL QUADRATIC FORMS [Ref. 31]

Let

$$\underline{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

be a Gaussian vector with

$$E\underline{Y} = \underline{A} \qquad \text{and} \qquad \underline{K} = E(\underline{Y}-\underline{A})'(\underline{Y}-\underline{A})$$

A. $Q = \underline{Y}'\underline{W}\,\underline{Y}$

Let $Q = \underline{Y}'\underline{W}\,\underline{Y}$ be the quadratic form where $\underline{W}$ is an $n \times n$ matrix. Let

$$\{\lambda_k\}_{k=1}^{n}$$

be the eignevalues of $\underline{W}\,\underline{K}$.

1. $E\underline{Y} = \underline{0}$ (zero mean)

$$\psi_Q(\mu) = |\underline{I} - 2i\mu\,\underline{W}\,\underline{K}|^{-1/2} = \left(\frac{1}{\prod_{k=1}^{n} (1 - 2i\mu\lambda_k)} \right)^{1/2}$$

$$EQ = \sum_{k=1}^{n} \lambda_k$$

$$\text{Var } Q = 2 \sum_{k=1}^{n} \lambda_k^2$$

2. $\underline{E\underline{Y} = \underline{A}}$

$$\psi_Q(\mu) = |\underline{I} - 2i\mu\, \underline{W}\, \underline{K}|^{-1/2} \exp\left(- \frac{1}{2} \underline{A}'\underline{K}^{-1} [\underline{I} - (\underline{I} - 2i\mu\, \underline{W}\, \underline{K})^{-1}]\underline{A}\right)$$

B. $Q = \underline{Y}'\underline{W}\,\underline{Y} + \underline{B}'\underline{Y}$

Let $Q = \underline{Y}'\underline{W}\,\underline{Y} + \underline{B}'\underline{Y}$; $\underline{B}' = n \times 1$ column vector. $\underline{W}$ is an $n \times n$ matrix.

1. $\underline{E\underline{Y} = \underline{0}}$ (zero mean)

$$\psi_Q(\mu) = |\underline{I} - 2i\mu\, \underline{W}\, \underline{K}|^{-1/2} \exp\left(- \frac{\mu^2}{2} \underline{B}'\underline{K}\, (\underline{I} - 2i\mu\, \underline{W}\, \underline{K})^{-1}\, \underline{B}\right)$$

2. $\underline{E\underline{Y} = \underline{A}}$

$$\psi_Q(\mu) = |\underline{I} - 2i\mu\, \underline{W}\, \underline{K}|^{-1/2} \exp\left(i\mu(\underline{A}'\underline{W}\,\underline{A} + \underline{B}'\underline{A})\right)$$

$$\cdot \exp\left(- \frac{\mu^2}{2} (2\underline{W}\,\underline{A} + \underline{B})'\ \ \underline{K}(\underline{I} - 2i\mu\, \underline{W}\, \underline{K})^{-1}(2\underline{W}\,\underline{A} + \underline{B})\right)$$

APPENDIX A. MISCELLANEOUS FORMS

1. A Zero Mean Complex Quadratic Form [Ref. 2]

$\underline{X}$ and $\underline{Y}$ are complex zero mean Gaussian vectors. Hence $E\underline{X} = E\underline{Y} = \underline{0}$. Components of the same vector are independent as usual.

Given: $$Ex_k x_k^* = E|x_k|^2 = m_{xx} \qquad k = 1, \ldots, n$$

$$Ey_k y_k^* = E|y_k|^2 = m_{yy} \qquad k = 1, \ldots, n$$

$$Ex_k^* y_k = m_{xy}, \qquad Ex_k y_k^* = m_{xy}^* \qquad k = 1, \ldots, n$$

Let $$\underline{M} = \begin{bmatrix} m_{xx} & m_{xy} \\ m_{xy}^* & m_{yy} \end{bmatrix}$$

a,b be real and c complex.

Define

$$q_k = a|x_k|^2 + b|y_k|^2 + cx_k^* y_k + c^* x_k y_k^* \qquad k = 1, \ldots, n$$

$$\underline{P} = \underline{M} \begin{bmatrix} a & c \\ c^* & b \end{bmatrix} = \begin{bmatrix} p_{xx} & p_{xy} \\ p_{xy}^* & p_{yy} \end{bmatrix}$$

$$q = \sum_{k=1}^{n} q_k$$

$$\alpha = \left\{ \left[\frac{p_{xx} + p_{yy}}{2|\underline{P}|} \right]^2 - \frac{1}{|\underline{P}|} \right\}^{1/2} + \frac{p_{xx} + p_{yy}}{|\underline{P}|}$$

$$\beta = \left\{ \left[\frac{p_{xx} + p_{yy}}{2|\underline{P}|} \right]^2 - \frac{1}{|\underline{P}|} \right\}^{1/2} - \frac{p_{xx} + p_{yy}}{2|\underline{P}|}$$

a. $n = 1$

$$g(q) = \begin{cases} \dfrac{\alpha\beta}{\alpha+\beta}\, e^{\beta q} & q < 0 \\ \\ \dfrac{\alpha\beta}{\alpha+\beta}\, e^{-\alpha q} & q \geqq 0 \end{cases}$$

$$G(q) = \begin{cases} \dfrac{\alpha}{\alpha+\beta}\, e^{\beta q} & q < 0 \\ \\ 1 - \dfrac{\beta}{\alpha+\beta}\, e^{-\alpha q} & q \geqq 0 \end{cases}$$

b. n

$$g(q) = \begin{cases} e^{\beta q} \displaystyle\sum_{k=0}^{n-1} \frac{(\alpha\beta)^n}{(\alpha+\beta)^{n+k}} \binom{n+k-1}{k} \frac{(-q)^{n-k-1}}{(n-k-1)!} & q < 0 \\ \\ e^{-\alpha q} \displaystyle\sum_{k=0}^{n-1} \frac{(\alpha\beta)^n}{(\alpha+\beta)^{n+k}} \binom{n+k-1}{k} \frac{(-q)^{n-k-1}}{(n-k-1)!} & q \geqq 0 \end{cases}$$

$$G(q) = \begin{cases} \displaystyle\sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \frac{(\alpha\beta)^n}{(\alpha+\beta)^{n+k}} \binom{n+k-1}{k} \frac{(-q)^{n-k-j-1}}{(n-k-j-1)!\ \beta^{j+1}} e^{\beta q} & q < 0 \\ \displaystyle 1 - \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \frac{(\alpha\beta)^n}{(\alpha+\beta)^{n+k}} \binom{n+k-1}{k} \frac{(-1)^{n-k-1}\, q^{n-k-j-1}}{(n-k-j-1)!\ \alpha^{j+1}} e^{-\alpha q} & q \geq 0 \end{cases}$$

2. Rayleigh (+) Rayleigh (Independent) [Ref. 30]

$$\underline{X}^{(1)} \in N_2(\underline{0},1); \qquad \underline{X}^{(2)} \in N_2(\underline{0},1)$$

$$r_1 = \|\underline{X}^{(1)}\|, \qquad r_2 = \|\underline{X}^{(2)}\|$$

Let $$z = r_1 + r_2$$

$$f(z) = \exp\left(-\frac{z^2}{4}\right)\left[\left(\frac{z^2}{4} - \frac{1}{2}\right)\sqrt{\pi}\ \mathrm{erf}\left(\frac{z}{2}\right) + \frac{z}{2}\exp\left(-\frac{z^2}{4}\right)\right] \qquad z \geq 0$$

3. Gaussian (×) Rayleigh (Independent) [Ref. 30]

$$x \in N_1\left(0,\sigma_1^2\right), \qquad \underline{X} \in N_2\left(\underline{0},\sigma_2^2\right)$$

$$r = \|\underline{X}\|$$

Let $$z = xr$$

$$f(z) = \frac{1}{2\sigma_1\sigma_2} \exp\left(-\frac{|z|}{\sigma_1\sigma_2}\right)$$

4. Gaussian (×) Rayleigh (+) Gaussian (Independent)

$$x \in N_1(0,1), \qquad \underline{X} \in N_2(\underline{0},1), \qquad y \in N_1(0,\sigma^2)$$

$$r = \|\underline{X}\|$$

Let
$$w = xr + y$$

$$f(w) = \frac{\exp\left(-w + \frac{\sigma^2}{2}\right)}{2}\left\{1 - \frac{\operatorname{erf}\left[\frac{\sigma}{\sqrt{2}} - \frac{w}{\sqrt{2}\,\sigma}\right] + \operatorname{erf}\left[\frac{\sigma}{\sqrt{2}} + \frac{w}{\sqrt{2}\,\sigma}\right]}{2}\right\}$$

[Ref. 30]

5. Gaussian (+) Rayleigh (Independent)

$$x \in N_1\left(0,\sigma_1^2\right), \qquad \underline{X} \in N_2\left(\underline{0},\sigma_2^2\right)$$

$$r = \|\underline{X}\|$$

Let
$$z = x + r$$

$$f(z) = \frac{\sigma_1 \exp\left(-\frac{z^2}{2\sigma_1^2}\right)}{\sqrt{2\pi}\left(\sigma_2^2 + \sigma_1^2\right)} + \frac{z\sigma_2 \exp\left(-\frac{z^2\sigma_2^2}{2\left(\sigma_1^2 + \sigma_2^2\right)}\right)}{2\left(\sigma_1^2 + \sigma_2^2\right)^{3/2}}\left[1 + \operatorname{erf}\left(\frac{z\sigma_2}{\sigma_1\sqrt{2\left(\sigma_1^2 + \sigma_2^2\right)}}\right)\right]$$

[Ref. 30]

6. Gaussian (+) Double Rayleigh (Independent)

$$x \in N_1(0,\sigma^2), \qquad \underline{X} \in N_2(\underline{0},1)$$

$$r = \|\underline{X}\| \qquad \text{and} \qquad r^* = \begin{cases} r & \text{with probability } \frac{1}{2} \\ -r & \text{with probability } \frac{1}{2} \end{cases}$$

Let $$w = x + r^*$$

$$f(w) = \frac{\sigma \exp\left(-\frac{w^2}{2\sigma^2}\right)}{\sqrt{2\pi}\,(1+\sigma^2)} + \frac{|w| \exp\left(-\frac{w^2}{2(1+\sigma^2)}\right)}{2(1+\sigma^2)^{3/2}} \operatorname{erf}\left[\frac{|w|}{\sigma\sqrt{2(1+\sigma^2)}}\right]$$

[Ref. 30]

7. General Products (Indpendent)

a. $x_k \in N_1(0,1)$, $y_k \in N_1(0,1)$, $k = 1, \ldots, n$; all independent.

Let $$z = \prod_{k=1}^{n} \frac{x_k}{y_k}$$

1) $n = 1$

$$f(z) = \frac{1}{\pi(1 + z^2)}$$

2) $n = 2$

$$f(z) = \frac{1}{\pi^2(z^2 - 1)} \ln (z^2)$$

3) $n = 3$

$$f(z) = \frac{1}{2!\ \pi^3(z^2 + 1)} \{[\ln z^2]^2 + 1\}$$

4) n = 4

$$f(z) = \frac{1}{3!\ \pi^4(z^2 - 1)} \{[\ln z^2]^3 + 4[\ln z^2]\}$$

5) n = 5

$$f(z) = \frac{1}{4!\ \pi^5(z^2 + 1)} \{[\ln z^2]^4 + 10[\ln z^2] + 9\}$$

6) n

$$f(z) = \frac{1}{\pi^n[1 - (-1)^n z^{-1}]} \sum_{k=0}^{n-1} \frac{1}{(n-1-k)!\ k!} (\ln z)^{n-1-k} \frac{d^k}{ds^k} [s \csc s]^n \Bigg|_{s=0}$$

[Ref. 34]

b. Other Products and Discussion of the Mellin Transformation

See Ref. 34.

8. Instantaneous Frequency of Narrow Band Gaussian Noise

See Ref. 35.

APPENDIX B. TRANSCENDENTAL FUNCTIONS

1. Definitions and Special Forms of Transcendental Functions

a. Error Function

$$\mathrm{erf}\ (x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt; \qquad x \geqq 0$$

$$\mathrm{erfc}\ (x) = \frac{1}{2}\,[1 - \mathrm{erf}\ (x)] = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt; \qquad x \geqq 0$$

b. Modified Bessel Functions

First Kind

$$I_\alpha(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{\alpha+2k}}{k!\ \Gamma(\alpha+k+1)} \qquad x \geqq 0$$

$$I_n(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2k}}{k!\ (n+k)!} \qquad n \ \text{integer}$$

$$I_{-1/2}(x) = \left(\frac{1}{2\pi x}\right)^{1/2} (e^x + e^{-x})$$

$$I_{n+(1/2)}(x) = \frac{1}{\sqrt{2\pi x}} \left[e^x \sum_{k=0}^{n} \frac{(-1)^k\ (n+k)!}{k!\ (n-k)!\ (2x)^k} + (-1)^{n+1} \cdot e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\ (n-k)!\ (2x)^k} \right]$$

Second Kind

$$k_{\alpha}(x) = \int_0^{\infty} \exp(-x \cosh t) \cosh \alpha t \, dt \qquad x \geqq 0$$

$$k_{n+(1/2)}(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!\,(2x)^k} \qquad \begin{array}{l} n \text{ integer} \\ x \geqq 0 \end{array}$$

c. Marcum Q Functions

$$Q_m(\alpha,\beta) = \int_{\beta}^{\infty} t\left(\frac{t}{2}\right)^{m-1} \exp\left(-\frac{t^2+\alpha^2}{2}\right) I_{m-1}(\alpha t)\, dt \qquad \begin{array}{l} \alpha \geqq 0 \\ \beta \geqq 0 \end{array}$$

$$Q_m(0,\beta) = \exp\left(-\frac{\beta^2}{2}\right) \sum_{k=0}^{m-1} \frac{1}{k!}\left(\frac{\beta^2}{2}\right)^k$$

$$Q_m(\alpha,0) = 1$$

d. Bivariate Normal Function

$$V(\alpha,\beta) = \frac{1}{2\pi} \int_0^{\alpha} e^{-t^2/2}\, dt \int_0^{(\beta/\alpha)t} e^{-x^2/2}\, dx \qquad \begin{array}{l} \alpha \geqq 0 \\ \beta \geqq 0 \end{array}$$

e. Elliptically Normal Probability Function

$$\Lambda(\alpha,\beta) = \int_0^{\beta} e^{-t} I_0(\alpha t)\, dt \qquad \begin{array}{l} \alpha \geqq 0 \\ \beta \geqq 0 \end{array}$$

$$\Lambda(0,\beta) = 1 - e^{-\beta}$$

$$\Lambda(\alpha,\infty) = \frac{1}{\sqrt{1-\alpha^2}} \;; \qquad \Lambda(1,\beta) = \beta e^{-\beta}\left(I_0(\beta) + I_1(\beta)\right)$$

f. Gamma and Beta Functions

Gamma Functions

$$\Gamma(\alpha) = \int_0^\infty e^{-t}\, t^{\alpha-1}\, dt \qquad \alpha > 0$$

$$\Gamma(n) = (n-1)! = (n-1)(n-2)(n-3) \ldots (3)(2)(1) \qquad n \text{ integer}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}; \qquad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

Beta Functions

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}\,(1-t)^{\beta-1}\,dt; \qquad \begin{matrix}\alpha > 0\\ \beta > 0\end{matrix}$$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \begin{matrix}\alpha > 0\\ \beta > 0\end{matrix}$$

$$B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!} \qquad n,m \text{ integers}$$

g. Hypergeometric and Confluent Hypergeometric Functions

Confluent Hypergeometric Functions

$${}_1F_1(\alpha,\beta;x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k}\frac{x^k}{k!} \qquad (\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \qquad \beta \neq 0, -1, -2, \ldots$$

$$ {}_1F_1(\alpha,\beta;x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 e^{xt}\, t^{\alpha-1}\, (1-t)^{\beta-\alpha-1}\, dt \qquad \beta > \alpha $$

$$ {}_1F_1(\alpha,\alpha;x) = e^x $$

$$ {}_1F_1(m,n;x) = \frac{(n-1)!}{(m-1)!} \sum_{k=0}^{\infty} \frac{(m+k-1)!}{k!\,(n+k-1)!}\, x^k \qquad m,n \text{ integers} $$

$$ {}_1F_1(n+1,n;x) = \left(1 + \frac{x}{n}\right) e^x $$

$$ {}_1F_1(1,n+1;x) = \frac{n!}{x^n}\left[e^x - \sum_{k=0}^{n-1} \frac{x^k}{k!}\right] $$

<u>Hypergeometric Function</u>

$$ {}_2F_1(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}\, (1-t)^{\gamma-\beta-1}\, (1-tx)^{-\alpha}\, dt \qquad \alpha > \beta > 0 $$

h. Generalized Laguerre Polynomial

$$ L_n^{\alpha}(z) = \frac{e^z z^{-\alpha}}{n!} \frac{d^n}{dz^n}\left(e^z z^{n+\alpha}\right) = \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+1)}\, {}_1F_1(-n,\alpha+1,z) $$

[Ref. 21, p. 189]

$$ L_n^{\alpha}(z) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-z)^m}{m!} $$

i. Whittaker Functions

$$W_{\alpha,\beta}(x) = \frac{x^{\beta+(1/2)}\ e^{-x/2}}{\Gamma\left(\beta - \alpha + \frac{1}{2}\right)} \int_0^\infty e^{-xt}\ t^{\beta-\alpha-(1/2)}\ (1+t)^{\beta+\alpha-(1/2)}\ dt$$

$$\beta - \alpha + \frac{1}{2} > 0$$

For n,m integers

$$W_{(1-m-n)/2,(n+m)/2}(x) = \frac{e^{-x/2}}{x^{(n+m-1)/2}} \qquad x > 0$$

$$W_{(m+n-1)/2,(n+m)/2}(x) = x^{(m+n+1)/2}\ e^{-x/2} \sum_{k=0}^{n+m-1} \frac{(n+m-1)!}{(n+m-k-1)!\ x^{k+1}} \qquad x \geq 0$$

$$W_{-n/2,m+[(n-1)/2]}(x) = \frac{x^{m+(n/2)}\ e^{-x/2}}{(m+n-1)!} \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(m+n+k-1)!}{x^{m+n+k}} \qquad x > 0$$

$$W_{n/2,m+[(n-1)/2]}(x) = \frac{x^{m+(n/2)}\ e^{-x/2}}{(m-1)!} \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} \frac{(m+k-1)!}{x^{m+k}} \qquad x > 0$$

$$W_{n/2,m-[(n-1)/2]}(x) = \frac{x^{m-[(n-1)/2]}\ e^{-x/2}}{(m-n)!} \sum_{k=0}^{m} \binom{m}{k} \frac{(m-n+k)!}{x^{m-n+k}} \qquad x > 0,\ m \geq n$$

$$W_{-n/2,m-[(n-1)/2]}(x) = \frac{x^{m-[(n-1)/2]}\ e^{-x/2}}{m!} \sum_{k=0}^{m-n} \binom{m-n}{k} \frac{(m+k)!}{x^{m+k+1}} \qquad x > 0,\ m \geq n$$

2. Some Useful Relationships between Transcendental Functions

a. Marcum Q Function

1) $n = 1, \quad Q_1(\alpha,\beta) = Q(\alpha,\beta)$

$$Q(\alpha,\beta) = \exp\left(-\frac{\alpha^2+\beta^2}{2}\right)\sum_{k=0}^{\infty}\left(\frac{\alpha}{\beta}\right)^k I_k(\alpha\beta) \qquad \beta > \alpha > 0$$

$$Q(\alpha,\beta) = 1 - \exp\left(-\frac{\alpha^2+\beta^2}{2}\right)\sum_{k=0}^{\infty}\left(\frac{\beta}{\alpha}\right)^k I_k(\alpha\beta) \qquad \alpha > \beta$$

$$Q(\alpha,\alpha) = \frac{1}{2}\left[1 + e^{-\alpha^2} I_0(\alpha^2)\right]$$

$$Q(\alpha,\beta) + Q(\beta,\alpha) = 1 + \exp\left(-\frac{\alpha^2+\beta^2}{2}\right) I_0(\alpha\beta)$$

For $\alpha \gg \beta \gg 1$

$$Q(\alpha,\beta) \sim 1 - \frac{1}{\alpha-\beta}\sqrt{\frac{\beta}{2\pi\alpha}}\exp\left(-\frac{(\alpha-\beta)^2}{2}\right)$$

For $\beta \gg \alpha \gg 1$

$$Q(\alpha,\beta) \sim \frac{1}{\beta-\alpha}\sqrt{\frac{\alpha}{2\pi\beta}}\exp\left(-\frac{(\beta-\alpha)^2}{2}\right)$$

$$\int_0^{\gamma} e^{x/c}\, Q(\sqrt{2a}; \sqrt{2bx})\, dx = -c + \frac{bc^2}{bc-1} \exp\left(\frac{a}{bc-1}\right) \left\{1 - Q\left(\sqrt{\frac{2abc}{bc-1}}, \sqrt{\frac{2(bc-1)\gamma}{c}}\right)\right\}$$

$$+ ce^{\gamma/c}\, Q(\sqrt{2a}, \sqrt{2b\gamma}) \qquad \text{For} \quad a,b,c > 0$$

$$\frac{d}{dx} Q(\alpha,x) = -x \exp\left(-\frac{x^2+\alpha^2}{2}\right) I_0(\alpha x) \qquad x \geqq 0$$

$$\frac{d}{dx} Q(x,\beta) = \beta \exp\left(-\frac{x^2+\beta^2}{2}\right) I_1(\beta x) \qquad x \geqq 0$$

$$\exp\left(-\frac{x}{c} - c\right) \sum_{k=0}^{\infty} \frac{(c^2x)^{k/2}}{(k!)^2} W_{k/2,(k+1)/2}(x) = Q(\sqrt{2c}, \sqrt{2x}) \qquad x \geqq 0$$

2) m

$$Q_m(\alpha,\beta) = Q(\alpha,\beta) + \exp\left(-\frac{\alpha^2+\beta^2}{2}\right) \sum_{k=1}^{m-1} \left(\frac{\beta}{\alpha}\right)^k I_k(\alpha\beta)$$

$$\int_0^{\gamma} e^{x/c}\, Q_m(\sqrt{2a}; \sqrt{2bx})\, dx = -c + c\left(\frac{bc}{bc-1}\right)^m \exp\left(\frac{a}{bc-1}\right)$$

$$\cdot \left\{1 - Q_m\left(\sqrt{\frac{2abc}{bc-1}}; \sqrt{\frac{2(bc-1)\gamma}{c}}\right)\right\}$$

$$+ ce^{\gamma/c}\, Q_m(\sqrt{2a}, \sqrt{2b\gamma}) \qquad a,b,c > 0$$

$$x^{(m-1)/2} \exp\left(-\frac{x}{c} - c\right) \sum_{k=0}^{\infty} \frac{(c^2x)^{k/2}}{k!\,(m+k-1)!} W_{(m+k-1)/2,(m+k)/2}(x) = Q_m(\sqrt{2c}; \sqrt{2x})$$

$$x \geqq 0$$

b. Modified Bessel Functions

First Kind

$$\frac{d}{dx} I_\alpha(x) = \frac{I_{\alpha-1}(x) + I_{\alpha+1}(x)}{2}$$

$$\int x^n I_{n-1}(x)\, dx = x^n I_n(x)$$

$$\int x^{-n} I_{n+1}(x)\, dx = x^{-n} I_n(x)$$

$$I_n(x) = I_{-n}(x)$$

$$\int e^n I_0(x)\, dx = xe^x [I_0(x) - I_1(x)]$$

$$\int e^{-x} I_0(x)\, dx = xe^{-x} [I_0(x) + I_1(x)]$$

$$\int e^x I_1(x)\, dx = e^x [(1-x)I_0(x) + xI_1(x)]$$

$$\int e^{-x} I_1(x)\, dx = e^{-x} [(1+x)I_0(x) + xI_1(x)]$$

$$\int_0^\infty xe^{-\gamma x^2} I_\mu(\alpha x)\, I_\mu(\beta x)\, \alpha x = \frac{1}{2\gamma} \exp\left(\frac{\alpha^2 + \beta^2}{4\gamma}\right) I_\mu\left(\frac{\alpha\beta}{2\gamma}\right) \qquad \begin{matrix} \mu > -1 \\ \gamma > 0 \end{matrix}$$

Second Kind

$$\int x^n K_{n-1}(x)\, dx = -x^n K_n(x)$$

$$\int x^{-n} K_{n+1}(x)\, dx = -x^{-n} K_n(x)$$

$$\int_0^\infty e^{-ax} K_0(bx)\, dx = \frac{1}{\sqrt{b^2 - a^2}} \operatorname{arc\,cos} \frac{a}{b} \qquad b > |a|$$

$$\int_0^\infty x^{\mu-1} K_\nu(\alpha x)\, dx = \frac{2^{\mu-2}}{\alpha^\mu} \Gamma\left(\frac{\mu+\nu}{2}\right) \Gamma\left(\frac{\mu-\nu}{2}\right) \qquad \begin{matrix} \mu+\nu > 0 \\ \alpha > 0 \end{matrix}$$

c. Whittaker Function and Gamma Function

$$W_{0,\beta}(x) = \left(\frac{x}{\pi}\right)^{1/2} K_\beta\left(\frac{x}{2}\right) \qquad x \geqq 0$$

$$2^{\alpha-2}\, \Gamma\left(\frac{\alpha-1}{2}\right) \Gamma\left(\frac{\alpha}{2}\right) = \Gamma\left(\frac{1}{2}\right) \Gamma(\alpha-1)$$

d. Confluent Hypergeometric Function

$${}_1F_1(\alpha,\beta;x) = e^x\, {}_1F_1(\beta-\alpha,\beta;-x)$$

$${}_1F_1(\alpha,\alpha+1,-x) = \alpha x^{-\alpha} \int_0^x e^{-t}\, t^{\alpha-1}\, dt = z^{-\alpha}\, \Gamma(\alpha+1)\, I\left(\frac{x}{\sqrt{\alpha}}, \frac{1}{\alpha}\right) \qquad \alpha > 0$$

where $I(\cdot,\cdot)$ is the incomplete gamma function.

$$ {}_1F_1(1,a+1;x) = e^{-x}\, x^a\, \Gamma(a+1)\; I\left(\frac{z}{\sqrt{a}},\, a-1\right) \qquad a > 0 $$

$$ {}_1F_1\left(\frac{1}{2},\, \frac{3}{2},\, -x^2\right) = \frac{\sqrt{\pi}}{2x}\, \mathrm{erf}\ x \qquad x > 0 $$

$$ {}_1F_1(-n,1;x) = L_n(x) \quad \text{original Laguerre polynomial} $$

$$ {}_1F_1(-n,\alpha+1,x) = \frac{n!\ \Gamma(\alpha+1)}{\Gamma(\alpha+n+1)}\, L_n^{\alpha}(x) \quad \text{general Laguerre polynomial} $$

$$ {}_1F_1\left(\alpha+\frac{1}{2},2\alpha+1,x\right) = \frac{2^{2\alpha}\ \Gamma(\alpha+1)\ e^{x/2}}{x^{\alpha}}\, I_{\alpha}\left(\frac{x}{2}\right) $$

$$ {}_1F_1\left(\frac{1}{2},\, 1;\ -x\right) = e^{-x/2}\ I_0\left(\frac{x}{2}\right) $$

$$ {}_1F_1\left(\frac{1}{2},2;-x\right) = e^{-x/2}\left[I_0\left(\frac{x}{2}\right) + I_1\left(\frac{x}{2}\right)\right] $$

$$ {}_1F_1\left(-\frac{1}{2},1,-x\right) = e^{-x/2}\left[(1+x)\ I_0\left(\frac{x}{2}\right) + xI_1\left(\frac{x}{2}\right)\right] $$

$$ {}_1F_1\left(\frac{3}{2},1,-x\right) = e^{-x/2}\left[(1-x)\ I_0\left(\frac{x}{2}\right) + xI_1\left(\frac{x}{2}\right)\right] $$

$$ {}_1F_1\left(\frac{3}{2},2,-x\right) = e^{-x/2}\left[I_0\left(\frac{x}{2}\right) - I_1\left(\frac{x}{2}\right)\right] $$

For recursive relationships see Ref. 19.

$$ \int_0^{\infty} x^{\beta}\, e^{-ax^2}\, I_{\mu}(\gamma x)\, dx = \frac{1}{2a^{(\beta+u+1)/2}}\left(\frac{\gamma}{2}\right)^{\mu} \frac{\Gamma\left(\frac{\beta+\mu+1}{2}\right)}{\Gamma(\mu+1)}\ {}_1F_1\left(\frac{\beta+\mu+1}{2},\ \mu+1,\ \frac{\gamma^2}{4a}\right) $$

$$ a > 0, \quad \beta+\mu > -1 $$

e. Generalized Laguerre Polynomial

$$\frac{n!}{\Gamma(\alpha+n+1)} \int_0^\infty e^{-z} z^\alpha L_n^\alpha(z) L_k^\alpha(z)\, dz = \delta_{nk}$$

f. Special

$$\sum_{k=m}^{n} \binom{\ell+k}{k-m} (1+x)^{n-k} = \sum_{k=m}^{n} \binom{n+\ell+1}{k-m} x^{n-k}$$

where m, n, and ℓ are integers and $0 \leq m \leq n$.

[Ref. 29]

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